THE FIVE DIMENSIONAL POLYHEDRAL SCHOENFLIES THEOREM

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Introduction. The central result of this note is a proof that in the 5-sphere with its usual piecewise linear structure, a subpolyhedron homeomorphic to S^4 bounds two topological 5-cells. Some related facts about imbeddings of S^{n-1} in S^n are also included. Our information about such imbeddings is quite incomplete, so it may be appropriate, after making a few definitions, to summarize current knowledge with regard to this problem.

We shall adhere to the notation of [5] and [14]. An imbedding h of S^k in S^n will be called *flat* if $(S^n, h(S^k)) \approx (S^{n-k-1} \circ S^k, S^k)$; it will be called *weakly flat* if $S^n - h(S^k) \approx S^{n-k-1} \circ S^k - S^k$. According to classical results every imbedding of S^{n-1} in S^n is flat if $n \leqq 2$. M. Brown's recent characterization [4], [5] is that an $(n-1)$-sphere is flat in S^n if and only if it is locally flat.

A polyhedron will be called a *piecewise linear n-sphere* or *n-cell* if it is piecewise linearly equivalent to Bd σ^{n+1} or σ^n, respectively. A *combinatorial n-manifold* (*with boundary*) is an n-polyhedron in which the link of each k-simplex is a piecewise linear $(n-k-1)$-sphere (or a piecewise linear $(n-k-1)$-cell). A *star n-manifold* (*with boundary*) is an n-polyhedron in which the link of each k-simplex is a topological $(n-k-1)$-sphere (or a topological $(n-k-1)$-cell). When $n \leqq 4$, these last two notions are equivalent. Newman [12] showed that if Σ is an $(n-1)$-sphere which is a subpolyhedron of a star triangulation of S^n and Σ is itself a star manifold (under the induced triangulation), then Σ is flat. Alexander proved in [2] that if Σ is an $(n-1)$-sphere which is a subpolyhedron of a piecewise linear n-sphere and the closure of one of Σ's complementary domains is a piecewise linear n-cell, then the closure of the other complementary domain is a piecewise linear n-cell as well.[2]

This brings us to what may be termed the polyhedral Schoenflies conjecture: Suppose an $(n-1)$-sphere Σ is a subpolyhedron of the piecewise linear n-sphere; must Σ be flat? If $n=3$, from work of

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[2] M. H. A. Newman actually established this theorem by more complicated methods in a somewhat earlier series of papers; see especially, *On the foundations of combinatory analysis situs*, Proc. Roy. Acad. Amsterdam 29 (1926), 610–641.