A DUALITY IN INTEGRAL GEOMETRY; SOME GENERALIZATIONS OF THE RADON TRANSFORM[1]

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1. Introduction. It was proved by J. Radon in 1917 (see [26]) that a differentiable function f of compact support on a Euclidean space can be determined from the integrals of f over each hyperplane in the space. Whereas Radon was primarily concerned with the dimensions 2 and 3, the following formulation for an arbitrary Euclidean space $\mathbf{R}^n$ was given by John [23], [24]. If ω is a unit vector let $J(\omega, p)$ denote the integral of f over the hyperplane $\{x \in \mathbf{R}^n \mid \langle x, \omega \rangle = p\}$ where $\langle\ ,\ \rangle$ denotes the inner product. Then, if $d\omega$ denotes the surface element on the unit sphere $\Omega = S^{n-1}$, and Δ the Laplacian,

$$f(x) = \tfrac{1}{2}(2\pi i)^{1-n} \Delta_x^{(n-1)/2} \int_\Omega J(\omega, \langle \omega, x \rangle) d\omega \qquad (n \text{ odd}), \tag{1}$$

$$f(x) = (2\pi i)^{-n} \Delta_x^{(n-2)/2} \int_\Omega d\omega \int_{-\infty}^{\infty} \frac{dJ(\omega, p)}{p - \langle \omega, x \rangle} \qquad (n \text{ even}), \tag{2}$$

where in the last integral, the Cauchy principal value is taken.

Applications. The applications of these formulas are primarily based on the following property: Consider the integrand in the integral over Ω, say the function $J(\omega, \langle \omega, x \rangle)$ in (1). For a fixed ω this function $x \to J(\omega, \langle \omega, x \rangle)$ is a plane wave, that is a function which is constant on each member of a family of parallel hyperplanes. Aside from the Laplacian, formulas (1) and (2) give a continuous decomposition of f into plane waves. Since a plane wave only amounts to a function of one real variable (along the normal to the hyperplanes) the formulas (1) and (2) can sometimes reduce a problem for n real variables to a similar problem for one real variable. This principle has been used effectively on partial differential equations with constant coefficients (see Courant-Lax [2], Gelfand-Shapiro [10], John [24], Borovikov [1], Gårding [4]) and even for general elliptic equations (John [24]).

Generalizations. The above representation of a function by means of

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