A GENERALIZATION OF MORSE-SMALE INEQUALITIES[1]

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In this paper we obtain relations between simple periodic surfaces of a vector field on a closed manifold M^n, and the betti numbers of M^n. When X is a gradient vector field of a nondegenerate function on M, the simple periodic surfaces are the critical points of the function and our relations are the Morse Inequalities. For Morse-Smale dynamical systems, the simple periodic surfaces are the critical points and closed orbits and we obtain the inequalities of Smale. In this case, where the periodic surfaces are singularities and closed orbits, we are able to remove Smale's normal intersection condition and replace it by the much weaker condition: there are no cycles of orbits among the periodic surfaces. Consequently, in this context, the need for approximating gradient fields by Morse-Smale systems is eliminated. This is an announcement of the results; detailed proofs will appear elsewhere.

I. Periodic surfaces of vector fields.

DEFINITIONS. Let X be a C^∞ vector field on M. A periodic i surface of X is a submanifold of M, invariant under X which is homeomorphic to T^i, the i dimensional torus. T^0 is a point so a periodic zero surface is a critical point of X; a periodic one surface is a closed orbit and a periodic two surface is a two torus which is the union of trajectories of X.

A simple periodic i surface β of X, of index j is a periodic surface β of M satisfying:

There is a tubular neighborhood N of β wherein X is topologically equivalent to one of the vector fields Y_1, $Y_2 \in \mathfrak{X}(T^i \times R^{n-i})$ defined by

I. $Y_1(\theta, y) = (1, f_1(\theta, y), B_1 y + G_1(\theta_1, y))$ where

$$(\theta, y) = (\theta_1, \cdots, \theta_i, y_1, \cdots, y_{n-i}) \in T^i \times R^{n-i}\ (1, f_1(\theta, y)) \in T^i, f_1 \colon T^i \times R^{n-i} \to 1 \times T^{i-1} \subset T^i,$$

f_1 is C^∞, B_1 is a real $n-i$ matrix having no eigenvalue with zero real part and $G_1\colon T^1 \times R^{n-i} \to R^{n-i}$ is C^∞, quadratic in y and $G_1(\theta_1, 0) = 0$,

[1] This is part of my doctoral dissertation, submitted at Berkeley in June, 1963. I wish to thank Professor Diliberto, who was my thesis director and originally suggested this research.