TENSOR PRODUCT ANALYSIS OF PARTIAL DIFFERENCE EQUATIONS

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1. Introduction. This note presents a few of the results which we have obtained by applying a classical and fundamental idea to the analysis of certain partial difference equations. The idea is that certain multidimensional problems can be solved by solving a few one-dimensional problems—it is the basis of the classical method of separation of variables of mathematical physics. In the case of partial difference equations, this idea leads to *tensor product* analysis of the matrices involved.

With this approach we accomplish the following: (i) Explicit *exact* solutions of problems consisting of separable partial difference equations and boundary conditions are obtained, (ii) A stable algorithm is devised with which these exact solutions can be evaluated with less work than approximate solutions can be computed by overrelaxation techniques, (iii) A simple, direct analysis of certain alternating direction implicit (ADI) methods is carried out and, as a result, a simple explanation of the power of this method is given, (iv) A necessary and sufficient condition is found for commutativity of certain matrices which occur in ADI schemes.

2. Tensor products applied to elliptic and parabolic boundary value problems. The *tensor product* (Kronecker product, direct product) of two matrices $A = \{a_{ij}\}$ and $B = \{b_{kl}\}$, denoted by $A \otimes B$, can be written as a matrix in block partition form:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

A detailed account of properties of tensor products is given in [8]. Some of the elementary properties are:

$$(A+B)\otimes C = A\otimes C + B\otimes C, \qquad A\otimes(B+C) = A\otimes B + A\otimes C,$$
$$(A\otimes B)(C\otimes D) = AC\otimes BD, \qquad (A\otimes B)^{-1} = A^{-1}\otimes B^{-1}.$$

For brevity, we do not indicate explicitly the sizes of the matrices involved; we assume throughout that the sizes of matrices and vectors are compatible with the indicated operations.