RATE OF GROWTH OF HURWITZ ENTIRE FUNCTIONS AND INTEGER VALUED ENTIRE FUNCTIONS

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1. **Introduction.** A Hurwitz function is a function $f(z)$ with $f^{(n)}(0) = integer$ for $n=0, 1, 2, \cdots$. An integer valued function is a function $g(z)$ with $g(n)=integer$ for $n=0, 1, 2, \cdots$.

It is well known that every transcendental Hurwitz entire function and every transcendental integer valued entire function must be at least of exponential order, type 1 and log 2 respectively, which are the best possible values. (Example: $f(z)=e^z$, $g(z)=2^z$.)

Various improvements on these facts have been studied to a considerable extent. It is the purpose of this note to establish the precise dividing line for the growth of these entire functions below which one finds only polynomials.

2. **Hurwitz entire function and integer valued entire function.**

DEFINITION. Let

$$\begin{aligned} \phi(r) &= \max_n \frac{r^n}{\Gamma(n+1)} \qquad (r \geqq 0) \\ &= \exp(r - (\log r)/2 - (\log 2\pi)/2 + 1/(24r) + O(1/r^2)). \end{aligned} \tag{1}$$

THEOREM 1. *Let $\psi(r)$ be any increasing function such that for every N, there exists an r_N, so that $\psi(r) > r^N$ for all $r > r_N$, then there exists a nondenumerable set of transcendental Hurwitz entire functions which satisfy*

$$M(r) < \phi(r) + \psi(r) \tag{2}$$

for all $r > R$, where R is a suitable positive number depending only on $\psi(r)$. Here $M(r)$ is the maximum modulus of $f(z)$ at $|z| = r$.

THEOREM 2. *A Hurwitz entire function is a polynomial if*

$$M(r) < \phi(r) + r^N \tag{3}$$

for some N and all $r > r_0$.

THEOREM 3. *There exists a denumerable infinite set of transcendental integer valued entire functions which satisfy*

$$M(r) < 2^r - r^N \tag{4}$$

for any fixed N and all $r > r_0$.