AVERAGES OF CONTINUOUS FUNCTIONS ON COUNTABLE SPACES[1]

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Introduction. Let $X=\{x_1, x_2, \cdots\}$ be a countably infinite topological space; then the space $C^*(X)$ of all bounded real-valued continuous functions f may be regarded as a space of sequences $(f(x_1), f(x_2), \cdots)$. It is well known [7, p. 54] that no regular (Toeplitz) matrix can sum all bounded sequences. On the other hand, if $(x_1, x_2, \cdots)$ converges in X (to x_m), then every regular matrix sums all f in $C^*(X)$ (to $f(x_m)$).

The main result of this paper is that if a regular matrix sums all f in $C^*(X)$ then it sums f to $\sum \alpha_n f(x_n)$, for some absolutely convergent series $\sum \alpha_n$. We use this to show that no regular matrix can sum all of $C^*(X)$ if X is extremally disconnected (the closure of every open set is open). This extends a theorem of W. Rudin [6], which has an equivalent hypothesis (X is embeddable in the Stone-Čech compactification βN of a discrete space) and concludes that not all f in $C^*(X)$ are Cesàro summable.

For any continuous linear functional ϕ on $C^*(X)$ one has a ("Riesz") representation $\phi(f)=\int f d\mu$, where μ is a Radon measure on βX. Our main result is just that X supports μ; μ is forced to be atomic since X is countable. We show further that X has a subset T, the set of *heavy points*, such that the functionals we are concerned with correspond exactly to measures μ supported by T with $\mu(T)=1$. Our knowledge of T is limited; it will be summarized elsewhere.

1. Representation. It is well known [7, p. 57] that a matrix $A=(a_{ij})$ defines a regular summability method if and only if it satisfies the conditions (1) $\sum_j a_{ij}=1+o(1)$, (2) $\sum_j |a_{ij}|$ is uniformly bounded, and (3) for each j, $a_{ij}\to 0$.

For all the present results on real-valued functions, we may assume without loss of generality that *our topological spaces are completely regular*. Then each countable space X has a base of open-and-closed sets, and each $f\in C^*(X)$ is a uniform limit of linear combinations of characteristic functions of these basic sets.

Suppose that A is a regular matrix such that $\phi_A(f)=\lim_{i\to\infty} \sum_j a_{ij}f(x_j)$ exists for each $f\in C^*(X)$. For each open-and-closed subset U of X, let c_U denote its characteristic function, and let

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