Fourier analysis on groups. By Walter Rudin. Interscience Tracts in Pure and Applied Math., No. 12. Wiley, New York, 1962. 9+285 pp. $9.50.

This work has come in season.

First, it fills a gap between "classical" harmonic analysis (as represented, for example, in Zygmund's book) and "abstract" harmonic analysis (as in Loomis's book). Fourier analysis on (abelian) groups introduces itself in a natural way if one wants to unify the methods which are used in studying ordinary trigonometric series and Fourier integrals, Fourier series or integrals with several variables, almost periodic functions, Taylor and Dirichlet series. It is still more important if one wants to look at the general concepts of Banach algebras, like homomorphisms, symbolic calculus, structure of ideals or subalgebras, in the very field from which the theory originated: the convolution algebra of summable functions on a locally compact abelian group G, the convolution algebra of bounded measures on G, and the algebras of their Fourier transforms (respectively denoted $L^1(G)$, $M(G)$, $A(\Gamma)$, $B(\Gamma)$ through the book and through our review; Γ is the dual group of G). The primary concern of the book is to show which tools of analysis are useful for problems which arise from algebraic considerations.

A second reason to welcome the book is that it provides the reader with much new material. It deals mainly with problems whose solutions were not known five or six years ago. Following the order of the book, we find, after two introductory chapters:

(Chapter III) The problem of finding all idempotent measures on G, i.e. all idempotent elements in $M(G)$ or $B(\Gamma)$, or, equivalently, all sets S in Γ whose characteristic function belongs to $B(\Gamma)$. The problem was investigated by Helson (G = the circle T), Rudin ($G=T^n$), and finally P. J. Cohen. To be brief, the answer is that the only sets S are the obvious ones. But the proofs are not easy, and they are linked with some old and difficult problems like Littlewood's conjecture, that

$$\int_{-\pi}^{\pi} \left| e^{in_1x} + \cdots + e^{in_kx} \right| dx > c \log k$$

(c absolute constant; $n_1, \cdots, n_k$ integers).

(Chapter IV) The problem of homomorphisms of group algebras. Which are the homomorphic mappings from $L^1(G)$ to $L^1(G')$, or, more generally, from $L^1(G)$ to $M(G')$? In other words, which are the $Y \subset \Gamma'$ and the mappings ϕ: $Y \rightarrow \Gamma$ such that $f \in A(\Gamma)$ implies $f(\phi) \in B(\Gamma')$? The first study in this direction is due to Beurling and Helson