DUALITY THEOREMS FOR CONVEX FUNCTIONS

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Communicated by A. M. Gleason, September 19, 1963

Let F be a finite-dimensional real vector space. A *proper convex function* on F is an everywhere-defined function f such that $-\infty < f(x)$ for all x, $f(x) < \infty$ for at least one x, and

$$f(\lambda x_1 + (1-\lambda)x_2) \leqq \lambda f(x_1) + (1-\lambda)f(x_2)$$

for all x_1 and x_2 when $0 < \lambda < 1$. Its *effective domain* is the convex set $\operatorname{dom} f = \{x \mid f(x) < \infty\}$. Its *conjugate* [**2**; **3**; **6**; **7**] is the function f^* defined by

$$(1) \qquad f^*(x^*) = \sup\{(x, x^*) - f(x) \mid x \in F\} \qquad \text{for each } x^* \in F^*,$$

where F^* is the space of linear functionals on F. The conjugate function is proper convex on F^*, and is always lower semi-continuous. If f itself is l.s.c., then f coincides with the conjugate f^{**} of f^* (where F^{**} is identified with F). These facts and definitions have obvious analogs for concave functions, with "inf" replacing "sup" in (1).

Suppose f is l.s.c. proper convex on F and g is u.s.c. proper concave on F. If

$$\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset,$$

where $\operatorname{ri} C$ denotes the relative interior of a convex set C, then

$$\inf\{f(x) - g(x) \mid x \in F\} = \max\{g^*(x^*) - f^*(x^*) \mid x^* \in F^*\}.$$

This was proved by Fenchel [**3**, p. 108] (reproduced in [**5**, p. 228]). The purpose of this note is to announce the following more general fact.

THEOREM 1. *Let F and G be finite-dimensional partially-ordered real vector spaces in which the nonnegative cones $P(F)$ and $P(G)$ are polyhedral. Let A be a linear transformation from F to G. Let f be a proper convex function on F and let g be a proper concave function on G. If there exists at least one $x \in \operatorname{ri}(\operatorname{dom} f)$ such that $x \geqq 0$ and $Ax \geqq y$ for some $y \in \operatorname{ri}(\operatorname{dom} g)$, then*

$$(2) \qquad \begin{aligned} &\inf\{f(x) - g(y) \mid x \geqq 0,\ Ax \geqq y\} \\ &\qquad = \max\{g^*(y^*) - f^*(x^*) \mid y^* \geqq 0,\ A^*y^* \leqq x^*\}, \end{aligned}$$

where A^* is the adjoint of A.

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[1] The material in this note stems from the author's recent doctoral dissertation at Harvard. Support was provided under grant AF-AFOSR-62-348 at the Computation Center, Massachusetts Institute of Technology.