SPLINE INTERPOLATION AND BEST QUADRATURE FORMULAE

BY I. J. SCHOENBERG

Communicated by Felix Browder, October 9, 1963

1. The spline interpolation formula. A spline function $S(x)$, of degree $k(\geqq 0)$, having the knots

$$x_0 < x_1 < \cdots < x_n, \tag{1}$$

is by definition a function of the class C^{k-1} which reduces to a polynomial of degree not exceeding k in each of the $n+2$ intervals in which the points (1) divide the real axis. The function $S(x)$ is seen to depend linearly on $n+k+1$ parameters. In [5, Theorem 2, p. 258] are given the precise conditions under which we can interpolate uniquely by $S(x)$ arbitrarily given ordinates at $n+k+1$ points on the real axis.

For the remainder of this note we set $k=2m-1$ $(1\leqq m\leqq n)$ and single out from this family of spline functions the

CLASS Σ_m: *The class of spline functions $S(x)$ of degree $2m-1$, knots* (1), *and the additional property that $S(x)$ reduces to polynomials of degree not exceeding $m-1$ in each of the ranges $(-\infty, x_0)$ and $(x_n, +\infty)$.*

The restriction that $m\leqq n$ is essential, otherwise Σ_m reduces to π_{m-1} (here and below π_k denotes a generic polynomial of degree $\leqq k$, as well as their class). In a paper [1] soon to appear C. de Boor observes that [5, Theorem 2] implies the following interesting

THEOREM 1 (C. DE BOOR). *Given m $(1\leqq m\leqq n)$, the points* (1) *and also arbitrary reals y_i $(i=0, \cdots, n)$, then there is a unique $S(x)$ such that*

$$S(x_i) = y_i \qquad (i = 0, \cdots, n). \tag{2}$$

Let us now consider this interpolating spline function $S(x)$ in a given finite interval $a\leqq x\leqq b$ containing the points (1). Its particular interest is due to the following

THEOREM 2 (C. DE BOOR). *Let $f(x)\in C^{m-1}[a, b]$, having an absolutely continuous $f^{(m-1)}(x)$, and be such that*

$$f(x_i) = y_i \qquad (i = 0, \cdots, n).$$

If $S(x)$ denotes the interpolating spline function of Theorem 1 *then*

$$\int_a^b (f^{(m)}(x))^2 dx \geqq \int_a^b (S^{(m)}(x))^2 dx$$