SCATTERING THEORY

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1. Let H be a Hilbert space, $U(t)$ a group of unitary operators. A closed subspace D_+ of H will be called *outgoing* if it has the following properties:

(i) $U(t)D_+ \subset D_+$ for t positive.
(ii) $\bigcap_{t>0} U(t)D_+ = \{0\}$.
(iii) $\bigcup_{t<0} U(t)D_+$ dense in H.

A prototype of the above situation is when H is $L_2(-\infty, \infty; N)$, i.e., the space of square integrable functions on the whole real axis whose values lie in some accessory Hilbert space N, $U(t)$ is translation by t, and D_+ is $L_2(0, \infty; N)$.

THEOREM 1.[3] *If D_+ is outgoing for the group $U(t)$, then H can be represented isometrically as $L_2(-\infty, \infty; N)$ so that $U(t)$ is translation and D_+ is the space of functions with support on the positive reals. This representation is unique up to isomorphisms of N.*

We shall call this representation an *outgoing translation representation* of the group.

Taking the Fourier transform we obtain an *outgoing spectral representation* of the group $U(t)$, where elements of D_+ are represented as functions in $A_+(N)$, that is the Fourier transform of $L(0, \infty; N)$. According to the Paley-Wiener theorem $A_+(N)$ consists of boundary values of functions with values in N, analytic in the upper half-plane whose square integrals along lines $\operatorname{Im} z = \text{const}$ are uniformly bounded.

An incoming subspace D_- is defined similarly and an analogous representation theorem holds, D_- being represented by functions with support on the negative axis, that is, by $L_2(-\infty, 0; N_-)$. N_- and N are unitarily equivalent and will henceforth be identified. In the application to the wave equation there is a natural identification of N and N_-.

Let D_+ and D_- be outgoing and incoming subspaces respectively for the same unitary group, and suppose that D_+ and D_- are *orthogonal.* To each function $f \in H$ there are associated two functions k_- and k_+, the respective incoming and outgoing translation representa-

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[3] We were informed by Professor Sinai that he has obtained and used a similar theorem.