OPERATIONS ON BINARY RELATIONS AND THEIR APPLICATIONS[1]

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1. **Introduction.** As far as the existence of the smallest congruence of a given type on a semigroup is concerned, Kimura, Yamada and the author discussed special cases of semigroups in [7; 11] and identity conditions in [8]; the author argued implication conditions in [9]; Kimura generalized them to algebraic systems [4]. Also, Clifford and Preston interpreted these results of the principle of the maximal homomorphic image in [3]. Although the proof of existence was easily obtained, the problem of constructing the smallest congruence in the general case still remains. In this paper we define types of relations by means of semi-closure operations and discuss the existence of the smallest relation of a given type. In particular, if we provide semi-closure operations with the condition "join conservative," then we can explicitly state the method of construction of the smallest relation of a given type. This paper is a simplification, resystematization, and generalization of the theory in [9].

2. **General theory of operations on relations.** Let E be a set. A binary relation ρ is a subset of the product set $E\times E$. Let $\mathfrak{B}$ be a complete lattice composed of binary relations with respect to the usual inclusion relation $\subseteq$. For an arbitrary subset $\mathfrak{A}=\{\rho_\alpha;\ \alpha\in\Gamma\}$ of $\mathfrak{B}$, the join and meet are denoted by

$$\bigcup_{\alpha\in\Gamma} \rho_\alpha \quad \text{or} \quad J(\mathfrak{A}) \quad \text{and} \quad \bigcap_{\alpha\in\Gamma} \rho_\alpha \quad \text{or} \quad M(\mathfrak{A})$$

respectively. $\mathfrak{B}$ is not required to be the collection of all binary relations. Consider a unary operation P, i.e., a mapping of $\mathfrak{B}$ into itself: $\rho\rightarrow\rho P$. The inclusion relation with respect to the operations is defined as follows:

(2.1) Q includes P, i.e., $P \leqq Q$ means $\rho P \subseteq \rho Q$ for all $\rho \in \mathfrak{B}$,

and hence $P=Q$ iff $\rho P=\rho Q$ for all $\rho\in\mathfrak{B}$. Accordingly, the join and meet, $\bigcup_\xi P_\xi$, $\bigcap_\xi P_\xi$, of a set $\{P_\xi;\ \xi\in\Xi\}$ are given as follows:

$$(2.2) \qquad \rho\Big(\bigcup_\xi P_\xi\Big) = \bigcup_\xi \rho P_\xi, \qquad \rho\Big(\bigcap_\xi P_\xi\Big) = \bigcap_\xi \rho P_\xi.$$

[1] This paper is a rapid report without proof. The proofs will appear in [10].