ON SEMIGROUPS IN ANALYSIS AND GEOMETRY

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Introduction. The investigations which I wish to report are concerned with certain transformation semigroups, particularly with extensions of some classical groups to semigroups. The concept "transformation semigroup" will be used in the following sense: let V be a manifold and let $\mathfrak{S}$ be a set of local homeomorphisms in V, i.e., homeomorphisms between two domains (open connected sets) in V. We call $\mathfrak{S}$ a semigroup if the following conditions (a) to (d) are satisfied.

(a) If $f \in \mathfrak{S}$ maps O_1 onto O_2 and $g \in \mathfrak{S}$ maps the domain O_2 onto O_3, then the composite mapping $g \circ f$, which transforms O_1 onto O_3, belongs to $\mathfrak{S}$ also.

(b) If O_1 is any subdomain of the domain O of $f \in \mathfrak{S}$, then f restricted to O_2 belongs to $\mathfrak{S}$ also.

(c) The identity map of V belongs to $\mathfrak{S}$.

(d) If a sequence of mappings $f_n \in \mathfrak{S}$, all defined in the same domain O, converges uniformly in any compact part of O and the limit f is again a homeomorphism of O, then f belongs to $\mathfrak{S}$ also.

It would conform better to the usual terminology to call $\mathfrak{S}$ a pseudo-semigroup. For the sake of simplicity of language we drop the prefix "pseudo" and speak of a semigroup *in* V instead. If all the maps of $\mathfrak{S}$ are restrictions of maps defined on the whole of V, we shall speak of a semigroup *on* V. In this case, only the maps on the whole of V need to be considered.

We shall say that a semigroup $\mathfrak{S}$ in V can be *characterized locally* if the further condition is satisfied.

(e) If the domain O is the union $\bigcup_\alpha O_\alpha$ of some domains O_α, and a homeomorphism f of O, restricted to any O_α, belongs to $\mathfrak{S}$, then f belongs to $\mathfrak{S}$ in the whole O.

If the inverse of any f of a semigroup $\mathfrak{S}$ in V also belongs to $\mathfrak{S}$, then we speak of a *group* in V. If, in particular, $\mathfrak{S}$ is a semigroup *on* V, the transformations in $\mathfrak{S}$ on the whole of V map V onto itself and form a topological group of V in the ordinary sense.

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