NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS[1]

BY FELIX E. BROWDER

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It is the object of the present note to present a new nonlinear version of the orthogonal projection method for proving the existence of solutions of nonlinear elliptic boundary value problems. The key point in this method is the application of a new general theorem concerning the solvability of nonlinear functional equations in a reflexive Banach space involving operators which may not be continuous. In several recent papers ([2], [3], [4], [5]) the writer obtained preliminary results in this direction involving operator equations in Hilbert space. The passage from Hilbert spaces to reflexive Banach spaces marks a tremendous increase in the power and applicability of this approach to nonlinear boundary value problems and involves a sharp development of its basic ideas.

We show the existence of variational solutions of elliptic boundary value problems for strongly elliptic systems of order $2m$ on a domain in R^n in generalized divergence form

$$Au = \sum_{|\alpha| \leq m} D^\alpha A_\alpha(x, u, \cdots, D^m u), \tag{1}$$

where the A_α are of polynomial growth in $(u, Du, \cdots, D^m u)$. Earlier results for equations of the form (1) were obtained in 1961–1962 by M. I. Visik ([9], [10], [11]) by a more concrete analytic approach under much stronger hypotheses than those applied in our basic existence theorem, Theorem 1 below. The result of Theorem 1 is both simpler and considerably more general than the results of Visik in the papers cited above.

Because of the potential wide applicability of our method for other nonlinear problems as well as its simplicity, we give the complete proof below.

1. Let Ω be an open subset of the Euclidean space R^n, where for convenience we assume Ω to be bounded and smoothly bounded. We denote the points of Ω by $x = (x_1, \cdots, x_n)$ and $\int f(x)dx$ denotes integration with respect to Lebesgue n-measure. We set

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j} \quad \text{for } 1 \leq j \leq n,$$

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