INVARIANT SUBSPACES OF CERTAIN LINEAR OPERATORS

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Communicated by Edwin Hewitt, August 5, 1963

1. Following the investigations of Pontrjagin [6] and Iohvidov [3] on linear operators in a Hilbert space with an indefinite inner product, M. G. Kreĭn [5] proved the following theorem.

THEOREM (PONTRJAGIN-IOHVIDOV-KREĬN). *Let E be the Hilbert space of infinite complex sequences $x=\{x_i\}$ with convergent $\sum_{i=1}^{\infty}|x_i|^2$, with norm $\|x\|=(\sum_{i=1}^{\infty}|x_i|^2)^{1/2}$. Let n be a positive integer and let*

$$J_n(x) = \sum_{i=1}^{n} |x_i|^2 - \sum_{i=n+1}^{\infty} |x_i|^2$$

for $x=\{x_i\}\in E$. If a linear transformation $\phi: E\to E$ is continuous in the norm topology, and if

(1) *$J_n(x)\geqq 0$ implies $J_n(\phi(x))\geqq J_n(x)$,*

then there exists an n-dimensional linear subspace F of E such that: (i) $\phi(F)\subset F$; (ii) $J_n(x)\geqq 0$ *for* $x\in F$; (iii) *every eigenvalue of the restriction of ϕ on F is of absolute value $\geqq 1$.*

This theorem is stronger than a result which Iohvidov [3] derived from the fundamental theorem of [6]. Iohvidov's theorem is so related to Pontrjagin's fundamental theorem that either one can be obtained from the other by a transform analogous to the Cayley transform (see [4]). Pontrjagin's proof of his theorem uses delicate and rather complicated arguments. Kreĭn's proof of the theorem stated above is much simpler and consists of an ingenious application of the fixed point principle.

In the present note we shall prove two results similar to the Pontrjagin-Iohvidov-Kreĭn theorem but of much more general nature, on existence of invariant subspaces of certain linear operators. It will be seen that the Pontrjagin-Iohvidov-Kreĭn theorem can be derived from our Theorem 2. All topological vector spaces considered here are implicitly assumed to be real or complex topological vector spaces satisfying the Hausdorff separation axiom.

2. We shall need the following lemma which was proved in [2].

LEMMA. *Let X be a nonempty compact convex set in a topological vector*

[1] This work was supported in part by the National Science Foundation, Grant G-24865.