MINIMAL COMMUTATIVE DIVISIBLE SEMIGROUPS

BY TAKAYUKI TAMURA

Communicated by Edwin Hewitt, May 22, 1963

1. **Introduction.** Let S be a commutative divisible semigroup whose binary operation is denoted by $+$. If for any element x of S and for any positive integer n, there is an element $y \in S$ such that

$$x = n \cdot y = y + \cdots + y, \qquad n \text{ times}$$

then S is said to be divisible. If y is unique then S is called uniquely divisible. For example the additive semigroup R of all positive rational numbers is a uniquely divisible semigroup, while the additive group $R/(1)$ of all positive rational numbers mod 1 is divisible but not uniquely divisible.

In this note we report some results on commutative divisible semigroups, especially minimal commutative divisible semigroups, without proof. The proof of the theorems will be given in another paper [6]. Throughout this paper any semigroup is assumed to be commutative, and "commutative" will be often omitted.

2. **Fundamental theorems.** The following basic propositions are used for the discussion in this paper.

(2.1) Any homomorphic image of a divisible semigroup is divisible.

(2.2) If S_i $(i=1, \cdots, k)$ are divisible (uniquely divisible) then the direct product of S_i is also divisible (uniquely divisible).

Suppose a system of divisible semigroups S_α $(\alpha \in \Lambda)$ is given. S_α^0 denotes the semigroup S_α with a two-sided identity 0 adjoined: $x+0=0+x=x$ for all $x \in S_\alpha$. The semigroup obtained as the discrete direct product $\prod_{\alpha\in\Lambda} S_\alpha^0$ excluding the identity is called the annexed product of S_α, and it is denoted by $\tilde{\prod}_{\alpha\in\Lambda} S_\alpha$.

(2.3) If S_α is divisible (uniquely divisible) then $\tilde{\prod}_{\alpha\in\Lambda} S_\alpha$ is also divisible (uniquely divisible).

A commutative semigroup S is called power-cancellative if $n \cdot x = n \cdot y$ implies $x=y$ for every n.

(2.4) Any power-cancellative semigroup S can be embedded into the smallest uniquely divisible commutative semigroup T in the following sense: If S is embedded into a uniquely divisible semigroup U, then T is embedded into U.

This was obtained by Hancock [2] and also by the author ([4] or [5]) independently.

For example the additive semigroup I of all positive integers is