INFINITE-DIMENSIONAL GROUP REPRESENTATIONS[1]

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1. Introduction. Let S be a set in which there is given a σ-field of "Borel sets" and let G be a topological group. Let sx be defined for all s in S and all x in G in such a manner that

(a) $sx_1x_2=(sx_1)x_2$,

(b) $se=s$,

(c) $s, x\to sx$ is a Borel function

from $S\times G$ to S. Under these circumstances we shall say that S is a Borel G space. We shall usually assume that S is *standard* in the sense that it is isomorphic as a Borel space to a Borel subset of a separable complete metric space. Let μ be a Borel measure in S, that is, a σ-finite countably additive measure defined on all Borel subsets of S. If $\mu(Ex)=\mu(E)$ for all E we shall say that μ is invariant. Given an invariant μ in the Borel space S we may form the Hilbert space $\mathfrak{L}^2(S, \mu)$ and observe that for each x in G the operator L_x such that $(L_x(f))(s)=f(sx)$ is a unitary operator. Moreover $x\to L_x$ is a unitary representation of the group G in a sense to be made precise below. When $S=G$, G is locally compact and sx is group multiplication, the measure μ exists and is essentially unique. The unitary representation L in this case is called the *regular representation* of G.

Consider the special case in which G is the compact group K of all rotations in the plane—or equivalently the group obtained from the additive group of the real line by identifying numbers which differ by integral multiples of 2π.

The functions e^{inx} $(n=0, \pm 1, \pm 2, \cdots)$ form a basis for $\mathfrak{L}^2(K)$ and each member of this basis generates a one-dimensional subspace which is invariant under each L_x. We have, in a sense to be defined below, a decomposition of the regular representation of K into ir-

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