A MINIMAL MODEL FOR SET THEORY

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In the proof of the consistency of the Continuum Hypothesis and the Axiom of Choice with the other axioms of set theory, Gödel [1] introduced the notion of a constructible set and showed that the constructible sets form a model for set theory. These sets are intuitively those which can be reached by means of a transfinite sequence of several simple operations. He then showed that the Axiom of Choice and Continuum Hypothesis held in the collection of constructible sets. If the original universe of sets is sufficiently rich in ordinal numbers, it will follow that every set is constructible, in which case we say that the Axiom of Constructibility is satisfied. This axiom implies the two axioms previously mentioned. However, from one point of view it may seem that this notion of constructibility does not intuitively correspond to what is meant by constructive since it may happen that all sets in the universe are constructive. In this paper we show that a more restricted notion of "construction" will yield a class of sets which form a minimal model for set theory. In this manner we prove the consistency of a stronger form of the Axiom of Constructibility. We observe that the idea of a minimal collection of objects satisfying certain axioms is well known in mathematics, for example, in group theory one often considers the subgroup generated by a collection of elements, and in measure theory we define the Borel sets as the smallest σ-algebra of sets containing the open sets.

We shall work within the framework of Zermelo-Frankel set theory (denoted by Z-F set theory); the characteristic feature of this theory is that the axiom of substitution consists of a countable number of statements, one for each definable relation $R(x, y)$, which say that if for some fixed set A and for all x in A there exists a unique y such that $R(x, y)$ holds, then there exists a set B consisting of precisely those y. Since much of the proofs of the theorems we state follow quite closely the arguments of [1], we shall be rather brief.

Our main result is

THEOREM 1. *There exists a collection of sets which satisfy Z-F set theory and such that any other such collection contains a sub-collection isomorphic to it.*

Here, we mean that the $\in$-relation is taken to be the usual one. By

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