UNIQUENESS, STABILITY AND ERROR ESTIMATION

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As in the previous note [1] B is a bounded region, $x=(x_i)\in B$, and subscripts on u, v, w, c denote partial differentiation. We assume u, v, w, $c\in C^{(2)}$ in B and continuous in $\bar{B}$, although these conditions are unnecessarily restrictive.[1] The normal derivative u_ν is as in [1], $p=+$ or $p=-$, ϵ^p and δ^p are nonnegative constants, and $\| \ \|$ denotes the Euclidean norm. Conditions with T are for $x\in B$, those with R for $x\in\partial B$.

As an introduction to the problems we shall consider, let

$$Tu = a(x)u + \sum a_i(x)u_i - \sum a_{ij}(x)u_{ij}, \qquad Ru = u - k(x)u_\nu$$

where $a(x)\geqq A$, $|a_1(x)|\leqq A_1$, $a_{11}(x)=1$, $[a_{ij}(x)]\geqq 0$, $0\leqq k(x)\leqq K$, and $|x_1|\leqq b$. Suppose $y=y(x_1)$ satisfies, for $|x_1|\leqq b$:

$$E = \inf(Ay - A_1|y'| - y'') > 0, \qquad D = \inf(y - K|y'|) > 0.$$

Then the inequalities

$$(1) \qquad p(Tu - Tv) \leqq \epsilon^p \quad \text{and} \quad p(Ru - Rv) \leqq \delta^p$$

imply $p(u-v)\leqq y(x_1)\max(\epsilon^p/E,\ \delta^p/D)$. The condition on a_1 can be replaced by $a_1\geqq -A_1$ if $y'\geqq 0$ and by $a_1\leqq A_1$ if $y'\leqq 0$.

The usefulness of this result and the ease of its proof suggest that it be extended to nonlinear problems, and that is our purpose. We first consider the operator and continuity conditions:

$$(2) \qquad \begin{aligned} Tu &= a(x,u,u_i) - \sum a_{ij}(x,u_k)u_{ij}, \\ \|a_{ij}(x,u_k) - a_{ij}(x,v_k)\| &\leqq A_1\|u_k - v_k\|, \\ p[a(x,u,u_i) - a(x,v,v_i)] &\geqq -A_1\|u_i - v_i\| \text{ for } p(u-v) > 0. \end{aligned}$$

Usually these hold only for u and v, but in Theorem 1 they hold for every pair of solutions, the constants depending on the pair:

THEOREM 1. *Let T be as in (2) and let $Ru=k_1(x,u)-k_2(x,u_\nu)$ where k_2 is nondecreasing and k_1 is strictly increasing in the last argument. Suppose the problem $Tw=\tau(x)$, $Rw=\rho(x)$ has solutions of the following two types:*

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[1] The main condition really needed is that $p(u-v)$ be upper semicontinuous in the closure of the set where $p(u-v)>0$.