THE PERMANENT ANALOGUE OF THE HADAMARD DETERMINANT THEOREM

BY MARVIN MARCUS

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1. **Statement of results.** In [2; 3] it was conjectured that if $A=(a_{ij})$ is an n-square positive semi-definite hermitian matrix then

$$\text{per } A \geqq \prod_{i=1}^{n} a_{ii}. \tag{1}$$

Here per A denotes the permanent of A: per $A = \sum_\sigma \prod_{i=1}^n a_{i\sigma(i)}$ where the summation is over the whole symmetric group of degree n. It was announced [1] and later proved [2] that per $(A) \geqq \det A$ and the Hadamard determinant theorem suggests that the product of the main diagonal entries of A in fact separates the permanent and the determinant of A. In this note we sketch a proof of an inequality that is substantially stronger than (1). Let $A(i)$ denote the principal submatrix of A obtained by deleting row and column i.

THEOREM. *If A is an $(r+1)$-square positive semi-definite hermitian matrix then*

$$(r+1)a_{11} \text{ per } A(1) \geqq \text{per } A \geqq a_{11} \text{ per } A(1). \tag{2}$$

If A has a zero row then (2) *is equality throughout. If A has no zero row then the lower equality holds if and only if $A=a_{11}\dotplus A(1)$; the upper equality holds if and only if A is of rank* 1.

We remark that what is true for $A(1)$ is true for any $A(i)$ because the permanent is unaltered by permutation of the rows and columns.

By an obvious induction on r we have the

COROLLARY. *If A is an n-square positive semi-definite hermitian matrix then*

$$\text{per } A \geqq \prod_{i=1}^{n} a_{ii} \tag{3}$$

with equality if and only if A has a zero row or A is a diagonal matrix.

2. **Proof of theorem.** We outline the proof of the theorem. Let U be an n-dimensional unitary space with inner product (x, y). For $1 \leqq r \leqq n$ define $U^{(r)}$ to be the space of r-tensors on U; that is, $U^{(r)}$ is the dual space of the space of all multilinear complex valued func-