A HARNACK INEQUALITY FOR NONLINEAR EQUATIONS

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Recently Moser has obtained a Harnack inequality for linear divergence structure equations with $n > 2$ variables. In this note we indicate how a similar procedure can be used also for nonlinear equations; in fact the equations in question need not even satisfy the usual ellipticity conditions. As applications of our main result, we obtain, among other things, an a priori estimate for the Hölder continuity of solutions and the general asymptotic behavior of positive solutions at an infinite singularity.

Consider specifically equations of the form

$$\text{div } A(x, u, u_x) = B(x, u, u_x), \qquad x = (x_1, \cdots, x_n) \in D, \tag{1}$$

where D is a bounded open set in Euclidean n-space. In this equation A is a given vector function of x, u, u_x, B is a given scalar function of the same variables, and

$$\text{div } A = \sum_1^n \frac{\partial A_i}{\partial x_i}, \qquad u_x = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right).$$

The structure of equation (1) is determined by the functions $A(x, u, p)$ and $B(x, u, p)$. We assume that they are measurable in x and continuous in u and p, and that they satisfy inequalities of the form

$$\begin{aligned} |A| &\leqq a|p|^{\alpha-1} + b|u|^{\alpha-1} + e, \\ |B| &\leqq c|p|^{\alpha-1} + d|u|^{\alpha-1} + f, \\ p\cdot A &\geqq a^{-1}|p|^{\alpha} - d|u|^{\alpha} - g, \end{aligned} \tag{2}$$

for $x \in D$ and all values of u and p. Here α, $1 < \alpha < n$, is a fixed exponent, a is a positive constant, and b through g are measurable functions on D in the respective Lebesgue classes

$$b, e \in L_{n/(\alpha-1)}; \qquad c \in L_{n/(1-\epsilon)}; \qquad d, f, g \in L_{n/(\alpha-\epsilon)}, \tag{3}$$

ϵ being some positive number less than or equal to one. [We can also treat the case $\alpha = n$. The case $\alpha > n$, moreover, is somewhat easier and can be handled by means of Morrey's lemma. For simplicity and brevity of presentation we shall here restrict consideration to the range $1 < \alpha < n$, as indicated above.]

The generality of these assumptions requires that equation (1) be interpreted in a weak sense. Let $u = u(x)$ be a function having strong