A CONTINUOUS TIME ANALOGUE OF THE THEORY OF RECURRENT EVENTS

BY J. F. C. KINGMAN

Communicated by William Feller, December 6, 1962

The theory of recurrent events developed by Feller [2] finds one of its most important applications in the theory of discrete time Markov chains. The object of this note is to summarise a continuous time analogue of Feller's theory which can be applied in a similar way to continuous time Markov chains.

1. The following definition of a (discrete time) recurrent event is readily seen to be equivalent to that of Feller. A recurrent event on a probability space $(\Omega, \mathcal{A}, P)$ is a family $\mathcal{E} = \{E(n), n=1, 2, \cdots\}$ of $\mathcal{A}$-measurable subsets of Ω, with the property that, for all positive integers $n_1 < n_2 < \cdots < n_k$,

$$(1)\quad \begin{aligned} &P\{E(n_1)E(n_2) \cdots E(n_k)\} \\ &\qquad = P\{E(n_1)\}P\{E(n_2 - n_1) \cdots E(n_k - n_1)\}. \end{aligned}$$

It follows that the probability of any event determined by the $E(n)$ can be calculated from a knowledge of the numbers

$$(2)\qquad u_n = P\{E(n)\},$$

and thus much of the interest in the theory of recurrent events is centered on the "renewal sequence" $\{u_n\}$. Let us write $\mathcal{R}$ for the class of all renewal sequences.

Because the word "recurrent" has come to be used in a different sense in Markov chain theory, we shall avoid it, and use instead the term "regenerative" to describe the events to be considered here. Then the form of the definition (1) suggests the following continuous time analogue.

A *regenerative event* $\mathcal{E}$ on a probability space $(\Omega, \mathcal{A}, P)$ is a family of $\mathcal{A}$-measurable subsets $E(t)$ $(t>0)$ of Ω, having the property that, whenever real numbers t_j satisfy

$$(3)\qquad 0 < t_1 < t_2 < \cdots < t_k,$$

then

$$(4)\quad P\{E(t_1)E(t_2) \cdots E(t_k)\} = P\{E(t_1)\}P\{E(t_2 - t_1) \cdots E(t_k - t_1)\}.$$

The function $p(t)$ defined by

$$(5)\qquad p(t) = P\{E(t)\}$$