# VECTOR LATTICES OF SELF-ADJOINT OPERATORS

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1. **Introduction.** A number of authors have dealt with the relations between order and commutativity in operator algebras [**1**; **2**; **3**]. Their results, together with Kadison's characterization [**4**] of factors as anti-lattices, lead one to the belief that commutativity and lattice structure are synonomous concepts. A closer look at these phenomena, however, reveals a number of discrepancies, as we shall presently see from several examples.

We have therefore singled out a special kind of lattice structure and we proceed to show that if the operators in a linear space of self-adjoint (s.a.) operators have this structure, then they form a commuting set. As a consequence of this, we are able to place the results of [**1**; **2**; **3**] in a more coherent framework; the "commutativity-from-order" theorems in these three papers are very elementary (but non-obvious) consequences of our single result; the proof of our central theorem is quite geometrical and provides more insight into the mechanisms relating order and commutativity. Proofs will appear elsewhere.

2. **Order properties.** We first fix some notation. If $a$ is a s.a. operator (bounded, except in §5), we set $|a| = (a^2)^{1/2}$ and define $a \wedge b = \frac{1}{2}(a+b-|a-b|)$.

Now let $V$ be a linear subspace of the space of all s.a. operators and consider the following properties:

(A) If $a \in V$, then $|a| \in V$.

(L) The set $V^+$ of positive operators in $V$ lattice orders $V$.

(T) $|a+b| \leqq |a| + |b|$, for all $a, b \in V$.

(P) $a \wedge b \geqq 0$ for $a, b \in V^+$.

(R) For $0 \leqq z \leqq a+b$ with $a, b, z \in V^+$, there exist $u, v \in V$ with $z = u+v$, $0 \leqq u \leqq a$ and $0 \leqq v \leqq b$. (The Riesz decomposition property.)

(SQ) $a^2 \leqq b^2$ for $a, b \in V$ with $-b \leqq a \leqq b$.

(J) $\{a, b\} \geqq 0$ for $a, b \geqq 0$ in $V$ (here $\{a,b\} = \frac{1}{2}(ab+ba)$ is the *Jordan product*).

(O) $a^2 \leqq b^2$ for $a, b \in V$ with $0 \leqq a \leqq b$.

(J$^+$) $\{a, b\} \geqq 0$ for $a, b \in V$ with $0 \leqq a \leqq b$.

From elementary computations, we obtain

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