DIFFERENTIAL INQEUALITIES

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Let B denote a bounded region in Euclidean n-space, with boundary ∂B and closure $\bar{B}$. We write $P = x = (x_1, x_2, \cdots, x_n) \in \bar{B}$, $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$, and similarly for v, c and y. The normal derivative u_ν is understood in the sense of Walter, namely:

$$u_\nu(P_0) = \limsup\, [u(P_k) - u(P_0)]\, |P_k - P_0|^{-1}$$

where $P_k \in B$, $P_0 \in \partial B$, and $P_k \to P_0$ in such a way that

$$(P_k - P_0)\, |P_k - P_0|^{-1}$$

tends to a fixed vector, ν. We have $u = u(x)$, $v = v(x)$, and

$$Tu = \phi(x) - f(x, u, u_i, u_{ij}), \qquad x \in B,$$
$$Ru = \mathcal{H}(x) - k(x, u, u_\nu), \qquad x \in \partial B.$$

Independent variables are denoted by the letter s. The letter p means "+" or "−," and has the same meaning in hypothesis and conclusion. We suppose ϵ^p and δ^p to be nonnegative constants. The statement "$f(x, v, v_i, v_{ij} \uparrow)$ is monotone" means that

$$p[f(x, v, v_i, v_{ij}) - f(x, v, v_i, s_{ij})] \geqq 0$$

when the matrix $p[(v_{ij}) - (s_{ij})] \geqq 0$, $p = \pm$. Other assertions of monotony are interpreted similarly. We assume $u \in C^{(2)}$, $v \in C^{(2)}$ in B and $u \in C$, $v \in C$ in $\bar{B}$, although discontinuities can be allowed as in [2].

It is convenient to write $v' = (v, v_i, v_{ij})$, a vector of $1+n+n^2$ components, and similarly for u, s, and y. Also $f' = (f_u, f_{ui}, f_{ui_j})$ with argument (x, v') or (x, s'), as the case may be. Similarly, $k' = (k_u, k_{u_\nu})$. The statement "f' is continuous in the neighborhood of v" means that there is an $h > 0$ such that $f'(x, s')$ is continuous for $|s' - v'| < h$. Other statements of this kind are understood similarly.

THEOREM I. *Let $k(x, u\downarrow, u_\nu)$ be strictly monotone, let $k(x, v, v_\nu \uparrow)$ be monotone, and let f' be continuous in the neighborhood of v. Suppose further:*

(i) *$f(x, u\downarrow, u_i, u_{ij})$ is monotone, and $f(x, s, s_i, s_{ij}\uparrow)$ is monotone in the neighborhood of v.*

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