EXAMPLE OF A PROPER SUBGROUP OF S_∞ WHICH HAS A SET-TRANSITIVITY PROPERTY[1]

BY GERALD STOLLER

Communicated by A. M. Gleason, November 16, 1962

S. M. Ulam, on page 33 of his book, *A collection of mathematical problems*, poses the following question: Let G be a subgroup of S_∞ [the group of all permutations of the integers] with the property that for every two sets of integers of the same power whose complements are also of the same power, there exists a permutation g of G which transforms one set into the other. Is $G=S_\infty$ (Chevalley, von Neumann, et al.)?

The answer to this question is no!

We change the problem immaterially by taking S_∞ to be the group of all permutations of the natural numbers rather than the integers; this is helpful since all infinite subsets of the natural numbers are order-isomorphic. A subgroup G which is transitive (wherever possible) on the set of all subsets of the natural numbers is defined by means of a finiteness condition.

Let N be the set of natural numbers with the usual ordering. Consider the set G of all $\sigma \in S_\infty$ satisfying the condition:

(F) there exist $A_1, A_2, \cdots, A_k, B_1, B_2, \cdots, B_k$ subsets of N such that $\bigcup_{i=1}^k A_i = N = \bigcup_{i=1}^k B_i$ and in addition, for all i, $\sigma: A_i \to B_i$ is an order-isomorphism.

Call $\{(A_1, B_1), (A_2, B_2), \cdots, (A_k, B_k)\}$ a class of order-pairs for σ.

Let $\sigma, \tau \in G$ where $\{(A_1, B_1), \cdots, (A_k, B_k)\}$ & $\{(C_1, D_1), \cdots, (C_q, D_q)\}$ are classes of order-pairs for σ & τ respectively. It is easily seen that

$$\{(\sigma^{-1}[B_i \cap C_j], \tau[B_i \cap C_j]) : i \in \{1, \cdots, k\} \;\&\; j \in \{1, \cdots, q\}\}$$

is a class of order-pairs for $\tau\sigma$, so that $\tau\sigma \in G$. Also $\sigma^{-1} \in G$ since $\{(B_1, A_1), (B_2, A_2), \cdots, (B_k, A_k)\}$ is a class of order-pairs for σ^{-1}. Consequently (since G is obviously nonempty) G is a subgroup of S_∞.

That G has the property stated in the problem is clear since subsets of N having the same power are order-isomorphic. The (at most) two order-isomorphisms needed allow us to define an element of G as required.

An element ρ of S_∞ which reverses arbitrarily long strings of natural numbers cannot be in G. For example, ρ can be given by: $\rho(m) = (n+1)^2 - (m+1-n^2)$ where $n^2 \leqq m < (n+1)^2$. Suppose that ρ satis-

[1] Research supported by N.S.F. Graduate Fellowship Number 22113 (1962–1963).