VECTOR-VALUED DISTRIBUTIONS AND THE SPECTRAL THEOREM FOR SELFADJOINT OPERATORS IN HILBERT SPACE

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If A is a bounded linear operator (with spectrum S) in a Banach-space E, the functional calculus of Dunford-Gelfand-Taylor[1] can be characterized as a continuous homomorphism $f \to f(A)$ of the algebra $\mathcal{H}(S)$[2] of all holomorphic functions on S into the Banach algebra $\mathcal{L}(E)$ with the properties $f(A) = I$ if $f(z) \equiv 1$ and $f(A) = A$ if $f(z) \equiv z$. If E is a Hilbert space and $A = A^*$ a selfadjoint operator in E, there exist a much more general functional calculus $f \to f(A) = \int f(s) dE_s$, based on the spectral decomposition $A = \int s dE_s$ of A.

We will show how this extended calculus can be developed as a continuous extension of the analytic functional calculus in such a way, that we get a new and very natural proof of the spectral theorem for selfadjoint operators in Hilbert space.

1. The analytic functional calculus. If S is the spectrum of A the resolvent $R(s) = (A - sI)^{-1}$ is an operator valued, holomorphic function in the complement of S on the Riemann-sphere Ω, vanishing at ∞. If $f(z)$ is holomorphic on S (: = holomorphic on a neighborhood of S), there exists a contour C, separating S from all singularities of f. Then

$$(1) \quad f(A) = (2\pi i)^{-1} \int_C f(s) R(s) ds = (2\pi i)^{-1} \int_C f(s)(A - sI)^{-1} ds$$

is independent of C and $f \to f(A)$ is a homomorphism of the algebra $\mathcal{H}(S)$ of all such functions f into the algebra $\mathcal{L}(E)$ of bounded endomorphisms of E.

This homomorphism is uniquely determined by the properties:

(a) $f(z) \equiv 1 \curvearrowright f(A) = I$,

(b) $f(z) \equiv z \curvearrowright f(A) = A$,

(c) $f_n(z) \Rightarrow f(z)$ (uniformly in a certain neighborhood of S) $\curvearrowright$ $f_n(A) \Rightarrow f(A)$ (in the norm topology of $\mathcal{L}(E)$).

In certain cases the homomorphism can be extended to larger classes of functions. We will study the extension to the class of differentiable functions on the real line.

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[1] See Dunford-Schwartz [1].

[2] $\mathcal{H}(S)$ is a topological algebra with the natural topology defined in Köthe [2].