FREE AND DIRECT OBJECTS

BY Z. SEMADENI[1]

Communicated by Edwin Hewitt, July 18, 1962

1. General considerations. Let $\mathfrak{B}$ be a bicategory;[2] the following terms are supposed to be familiar to the reader: object, morphism (=map of the class in the question), equivalence (=isomorphism) injection, surjection (=projection in the sense of [**13**; **9**]). A morphism $\alpha: A \rightarrow B$ is called a *retraction* (and B is called a *retract* of A) if there exists a cross-section $\beta: B \rightarrow A$ i.e., a morphism such that $\alpha\beta$ is the identity $\epsilon_B: B \rightarrow B$. If this is the case, α must be a surjection and β must be an injection. Map (A, B) will denote the set of all morphisms $\alpha: A \rightarrow B$.

An object S will be called a *singleton* if Map(S, A) is not void and Map(A, S) consists of exactly one morphism for every object A; dually S is a *cosingleton* if Map$(A, S) \neq \varnothing$ and Map(S, A) consists of exactly one morphism for every A. All singletons and cosingletons are equivalent (if they exist). S is a singleton and a cosingleton simultaneously if and only if it is a null object. An example of a singleton which is not a null object is a one-point space in the category of topological spaces.

$\{A_t\}_{t \in T}$ being a set of objects, ΣA_t and ΠA_t will denote the free and direct join of it (cf. [**12**, §12]) with monomorphisms $\sigma_t: A_t \rightarrow \Sigma A_u$ and epimorphisms $\pi_t: \Pi A_u \rightarrow A_t$, respectively.

PROPOSITION 1. *If $\mathfrak{B}$ has a singleton or a cosingleton, then the monomorphisms $\sigma_t: A_t \rightarrow \Sigma A_u$ are injections admitting retractions $\pi_t: \Sigma A_u \rightarrow A_t$ and, dually, the epimorphisms $\pi_t: \Pi A_u \rightarrow A_t$ are surjections admitting cross-sections $\sigma_t: A_t \rightarrow \Pi A_u$.*

According to the standard definition an object P is *projective* if for every surjection $\alpha: A \rightarrow B$ and every $\beta: P \rightarrow B$ there exists $\gamma: P \rightarrow A$ such that $\alpha\gamma = \beta$, and I is *injective* if for every injection $\alpha: B \rightarrow A$ and every $\beta: B \rightarrow I$ there exists $\gamma: A \rightarrow I$ with $\gamma\alpha = \beta$.

PROPOSITION 2. *The retracts and free joins of projective objects are projective; the retracts and direct joins of injective objects are injective.*

An object M with be called a *coseparator* if for any two objects A and B and for any morphisms $\alpha: A \rightarrow B$ and $\beta: A \rightarrow B$, the condition $\alpha\gamma = \beta\gamma$ for all $\gamma \in$ Map(M, A) implies $\alpha = \beta$. Let us notice that any

[1] Research supported partially by the National Science Foundation.

[2] We assume Isbell's system of axioms, cf. [9], also [5; 7; 12; 13].