CLASS D SUPERMARTINGALES

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The decomposition of a continuous parameter supermartingale into the difference of a martingale and a process with increasing sample functions has been studied by Meyer [1]. Meyer has shown that a non-negative uniformly integrable right-continuous (i.e., right-continuous sample functions) supermartingale $\{y_t: 0 \leqq t \leqq +\infty\}$ with $\lim_{t\to\infty} y_t = 0$ can be decomposed in this manner if and only if the supermartingale is of class D on $[0, \infty]$. Such a supermartingale is of class D on $[0, \infty]$ if the family of random variables $\{y_T: T \in \mathfrak{J}\}$, where $\mathfrak{J}$ is the class of stopping times for the process, is uniformly integrable. Although Meyer has given some sufficient conditions under which a supermartingale is of class D on $[0, \infty]$, the existence of supermartingales which are uniformly integrable but not of class D on $[0, \infty]$ has not been settled. An example of a uniformly integrable supermartingale which is not of class D on $[0, \infty]$ is given below. A necessary and sufficient condition for a supermartingale to be of class D on $[0, \infty]$ is proven under the additional hypothesis that almost all sample functions of the process are continuous. This condition does not involve stopping times.

Let (Ω, F, P) denote a probability measure space. Points of Ω will be denoted by ω. If $\{y_t, F_t: 0 \leqq t \leqq \infty\}$ is a supermartingale, a stopping time is a non-negative random variable T, which may take on the value $+\infty$, such that $\{\omega: T(\omega) < t\} \in F_t$ for every t. The class of such stopping times will be denoted by $\mathfrak{J}$. It is well known that $S \wedge T$, the infimum of the two stopping times S and T, is a stopping time.

THEOREM. *Let $\{y_t, F_t: 0 \leqq t \leqq \infty\}$ be a non-negative right-continuous supermartingale. If the family $\{y_T: T \in \mathfrak{J}\}$ is uniformly integrable, then $\lim_{n\to\infty} nP[\sup_{0\leq t\leq\infty} y_t > n] = 0$. The converse holds if the supermartingale has sample functions which are continuous with probability* 1.

PROOF. Let $y^* = \sup_{0\leq t \leq \infty} y_t$ and let $T_n(\omega) = \inf\{t: y_t(\omega) \geqq n\}$ (the infimum of the empty set is defined to be $+\infty$). Then $T_n \in \mathfrak{J}$ for each positive integer n. For any $n \geqq \xi$,

$$\int_{y_{T_n} \geq \xi} y_{T_n} dP \geqq \int_{y_{T_n} \geq n} y_{T_n} dP \geqq nP[y_{T_n} \geqq n] \geqq nP[y^* > n].$$

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