A NOTE ON THE JACOBI THETA FORMULA

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In this note we show that Jacobi's identity [1, p. 280]

$$(1) \qquad \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}t)(1+q^{2n-1}t^{-1}) = \sum_{n=-\infty}^{\infty} q^{n^2}t^n$$

implies relations between various partition functions of two arguments, namely (5), (7) and (8) below.

In (1) take

$$q^2 = xy, \qquad t = xy^{-1}, \qquad |xy| < 1.$$

Then (1) becomes

$$(2) \quad \prod_{n=1}^{\infty}(1-x^ny^n)(1+x^ny^{n-1})(1+x^{n-1}y^n) = \sum_{n=-\infty}^{\infty} x^{n(n+1)/2}y^{n(n-1)/2}.$$

Let $\alpha(n, m)$ denote the number of partitions of (n, m) into distinct parts

$$(a, a-1), \qquad (b-1, b) \qquad (a, b = 1, 2, 3, \cdots),$$

so that we have the generating function

$$(3) \qquad \sum_{n,m=0}^{\infty} \alpha(n, m)x^ny^m = \prod_{n=1}^{\infty}(1+x^ny^{n-1})(1+x^{n-1}y^n).$$

Then by (2) and (3)

$$(4) \qquad \sum_{n,m=0}^{\infty} \alpha(n, m)x^ny^m = \prod_{n=1}^{\infty}(1-x^ny^n)^{-1} \sum_{r=-\infty}^{\infty} x^{r(r+1)/2}y^{r(r-1)/2}.$$

Since

$$\prod_{n=1}^{\infty}(1-x^ny^n)^{-1} = \sum_{n=0}^{\infty} p(n)x^ny^n,$$

where $p(n)$ is the number of unrestricted partitions of n, it follows from (4) that

$$(5) \qquad \alpha(n, m) = p\left(n - \frac{1}{2}(n-m)(n-m+1)\right).$$

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