THE CHARACTERIZATION OF FUNCTIONS ARISING AS POTENTIALS. II

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1. Statement of result. We continue our study of the function spaces L^p_α, begun in [7]. We recall that $f \in L^p_\alpha(E_n)$ when $f = K_\alpha * \phi$, where $\phi \in L^p(E_n)$. K_α is the Bessel kernel, characterized by its Fourier transform $K_\alpha(x)\hat{} = (1+|x|^2)^{-\alpha/2}$. It should also be recalled that the space L^p_k, $1 < p < \infty$, with k a positive integer, coincides with the space of functions which together with their derivatives up to and including order k belong to L^p; (see [2]).

It will be convenient to give the functions in L^p_α their strict definition. Thus we redefine them to have the value $(K_\alpha * \phi)(x)$ at every *point* where this convolution converges absolutely. With this done, and if $\alpha - (n-m)/p > 0$, then the restriction of an $f \in L^p_\alpha(E_n)$ to a fixed m-dimensional linear variety in E_n is well-defined (that is, it exists almost everywhere with respect to m-dimensional Euclidean measure). The problem that arises is of characterizing such restrictions.

The problem was previously solved in the following cases:

(i) When p is arbitrary, but $\alpha = 1$, in Gagliardo [3].

(ii) When $p = 2$, and α is otherwise arbitrary in Aronszajn and Smith [1]. In each case the solution may be expressed in terms of another function space, W^p_α, which consists of those $f \in L^p(E_n)$ for which the norm[2]

$$\|f\|_p + \left[\int_{E_n}\int_{E_n} \frac{|f(x-y) - f(x)|^p}{|y|^{n+\alpha p}}\, dxdy\right]^{1/p}$$

is finite, when $0 < \alpha < 1$. When $0 < \alpha < 2$, there is a similar definition of W^p_α (consistent with the previous one for $0 < \alpha < 1$) which replaces the difference $f(x-y) - f(x)$ by the second difference $f(x-y) + f(x+y) - 2f(x)$. Finally for general $\alpha \geq 2$, the spaces W^p_α are defined recurrently by $f \in W^p_\alpha$ when $f \in L^p$ and $\partial f/\partial x_n \in W^p_{\alpha-1}$, $k = 1, \cdots, n$.

In stating our result we let E_m denote a fixed proper m dimensional subspace of E_n, and Rf denote the restriction to E_m of a function defined on E_n.

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[2] Such norms were considered when $n = 1$ in [5]. The space is also considered in [6] and [9]; in the latter it is denoted by $\Lambda^{p,p}_\alpha$.