EVERY PLANAR GRAPH WITH NINE POINTS HAS A NONPLANAR COMPLEMENT

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In a *complete graph* every two points are joined by a line (are *adjacent*). A complete graph with n points is denoted by K_n. Let G be a graph with n points considered as a subgraph of K_n. The complement $\overline{G}$ of G is the graph obtained by removing all lines of G from K_n. The following problem was stated by Harary [2]: What is the least integer n such that every graph G with n points or its complement $\overline{G}$ is nonplanar? Harary [3] observed that $n \leqq 11$. It is readily seen that $n > 8$. In this note we shall outline the proof that $n = 9$, verifying a conjecture of J. L. Selfridge.

THEOREM. *If G is a graph with nine points, then G or its complement $\overline{G}$ is nonplanar.*

Let $p(G)$ be the number of points, $q(G)$ the number of lines, and $k(G)$ the number of components of graph G. Let $K_{m,n}$ be a graph with $m+n$ points, m points of one color and n points of another, in which two points are adjacent if and only if their colors are different. Kuratowski [5] proved the classic theorem that a graph is nonplanar if and only if it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.

PROPOSITION 1. *In each of the following cases, a graph G is nonplanar.*

(i) $p(G) \geqq 6$ *and* $k(\overline{G}) \geqq 4$.

(ii) $p(G) \geqq 7$, $k(\overline{G}) \geqq 3$ *and $\overline{G}$ has at most one isolated point.*

(iii) $p(G) \geqq 7$, $k(\overline{G}) = 2$ *and each component of $\overline{G}$ contains at least three points.*

(iv) $p(G) \geqq 9$ *and* $k(\overline{G}) \geqq 3$.

In each of these cases, it is easy to see that G contains $K_{3,3}$ as a subgraph. Thus G is nonplanar by Kuratowski's theorem.

PROPOSITION 2. *If $p(G) \geqq 9$, $k(\overline{G}) = 2$ and G is planar, then $\overline{G}$ is nonplanar.*

By Propositions 1 and 2, it is sufficient to prove the theorem under the hypothesis that $\overline{G}$ is connected.

Let G be a planar graph with $p(G) \geqq 4$. Imbed G into a 2-sphere S. By Fáry's theorem [1], there exists a triangulation T of S whose

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