AN EXPANSION FORMULA FOR DIFFERENTIAL EQUATIONS

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Denote by $z=(z^1, \cdots, z^n)$ the coordinates of the n-dimensional complex space. If $m=(m_1, \cdots, m_n)$ is an n-tuple of nonnegative integers, then we write $z^m=(z^1)^{m_1} \cdots (z^n)^{m_n}$. Any polycylinder mentioned in this paper will have its center at the origin.

Let $f(t, z)$ depend on the complex variables z and a parameter t over a measurable set I in a measure space with measure μ. We say that $f(t, z)$ is dominatedly integrable over I for z in a polycylinder U, if the following conditions are satisfied:

(a) For a.e. (almost every) value of the parameter t in I, $f(t, z)$ is holomorphic in U.

(b) The expansion $\sum a_m(t)z^m$ of $f(t, z)$ is such that each coefficient a_m is integrable over I.

(c) The series $\sum \int_I |a_m(t)| d\mu z^m$ converges in U.

It can be easily shown that $F(z)=\int_I f(t, z)d\mu$ is holomorphic in U, and this integration commutes with partial differentiation with respect to z.

DEFINITION. We say that $A(t)=\sum a^i(t, z)\partial/\partial z^i$, t being a real variable, is a t.d.i.t. (time dependent infinitesimal transformation) if there exists a polycylinder U such that, for z in U, each $a^i(t, z)$ is dominatedly integrable in the sense of Lebesgue over any finite interval I.

DEFINITION. For $f(z)$ holomorphic about the origin, we define $T^*(A; t, t_0)f$, or simply $T^*(t)f$, to be the sum function of the series

$$T_0^*(t)f + \cdots + T_r^*(t)f + \cdots,$$

where $T_0^*(t)f=f$ and, for $r>0$, $T_r^*(t)f=\int_{t_0}^t T_{r-1}^*(s)A(s)fds$.

Our main purpose is to prove the formula

$$(T^*(t)f)(z_0) = f(T(t)z_0), \tag{1}$$

where $z=T(t)z_0$ denotes the solution of the system of differential equations $dz^i/dt=a^i(t, z)$ with the initial condition $z(t_0)=z_0$.

By direct computation, it is verified that

$$\frac{d}{dt}\sum_{i=0}^{r} (T_i^*(t)f)(T_{r-i}^*(t)g) = dT_r^*(t)(fg)/dt.$$

Consequently, if $T^*(t)f$ and $T^*(t)g$ both converge absolutely in a