HOMOGENEOUS RIEMANNIAN MANIFOLDS OF NEGATIVE CURVATURE

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In [1], Wolf proved that a homogeneous Riemannian manifold of constant negative curvature is necessarily simply connected. The purpose of this paper is to prove the following generalization.

THEOREM. *Let M be a homogeneous Riemannian manifold possessing the following properties:*

(1) *The sectional curvature is nonpositive everywhere on M;*

(2) *For every nonzero tangent vector of M, there exists a tangent plane with negative sectional curvature which contains it.*

Then M is simply connected.

As an immediate consequence, we have

COROLLARY. *Let M^* be a simply connected Riemannian symmetric space whose components in the De Rham decomposition are all noncompact and non-Euclidean. If D is a properly discontinuous group of isometries of M^* such that $M = M^*/D$ is homogeneous, then D contains only the identity element.*

PROOF OF THEOREM. Let M^* be the universal covering space of M with the naturally induced Riemannian structure, so that $M = M^*/D$ where D is a properly discontinuous group of isometries acting freely on M^*. Let $p: M^* \to M$ be the covering projection. Let x be an arbitrary point of M. Assuming that D is nontrivial, let x_0^* and x_1^* be two distinct points of M^* such that $x = p(x_0^*) = p(x_1^*)$. Let $\tau^* = x_t^*$, $0 \leqq t \leqq 1$, be a geodesic from x_0^* to x_1^* where t is an affine parameter; τ^* is unique if M possesses the property (1). Let $\tau = x_t$, $0 \leqq t \leqq 1$, be the geodesic in M defined by $x_t = p(x_t^*)$ so that $x = x_0 = x_1$.

LEMMA. *If M is a homogeneous Riemannian manifold with the property (1), then τ is an orbit of a 1-parameter group of isometries so that the closed geodesic τ is smooth even at $x = x_0 = x_1$.*

PROOF OF LEMMA. Let X be an infinitesimal isometry of M tangent to τ at x_0. Then X is a Jacobi field along τ. At each point of τ we decompose X into a tangent vector and a normal vector to τ so that we have $X = Y + Z$ where Y (resp. Z) is a Jacobi field along τ tangent to τ (resp. normal to τ). Since Z vanishes at $x_0 = x_1$ and since the curvature of M is nonpositive, Z vanishes at every point of τ. This