UNIFORM APPROXIMATION AND MAXIMAL IDEAL SPACES

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Let X be a compact set in the z-plane. We are interested in two function spaces associated with X:

$C(X)$ = space of all continuous complex-valued functions on X.

$P(X)$ = space of all uniform limits of polynomials on X.

Thus a function f on X lies in $P(X)$ if there exists a sequence $\{P_n\}$ of polynomials converging to f uniformly on X.

Clearly $P(X)$ is part of $C(X)$.

QUESTION I. When is $P(X) = C(X)$, i.e. every continuous function approximable by polynomials?

QUESTION II. If $P(X) \neq C(X)$, how can we characterize those functions on X which lie in $P(X)$?

The first man to pay any attention to these problems I believe was Weierstrass who in 1885 showed $P(X) = C(X)$ when X is the unit interval on the real axis. If instead of the unit interval we consider an arbitrary Jordan arc (homeomorph of the unit interval), the problem is much harder. Walsh [1] proved

THEOREM. *For every Jordan arc J in the plane, $P(J) = C(J)$.*

Instead of a Jordan arc, consider now a simple closed Jordan curve Γ. It is easy to see that now $P(\Gamma) \neq C(\Gamma)$. The reason is this: suppose f is in $P(\Gamma)$. Then there is a sequence $\{P_n\}$ of polynomials tending to f uniformly on Γ. Hence $|P_n - P_m|$ tends to 0 uniformly on Γ. If z lies inside Γ,

$$| P_n(z) - P_m(z) | \leqq \max_{\Gamma} | P_n - P_m |,$$

by the maximum principle. Hence $P_n - P_m$ approaches 0 uniformly on W, the interior of Γ, whence P_n tends to a limit uniformly on W; call it F. Clearly F is analytic on W and has f as boundary values on Γ. Thus we have:

If f is in $P(\Gamma)$, then f is the boundary function of an analytic function. This rules out many f's in $C(\Gamma)$, e.g. any real-valued nonconstant f. This raises the question: Does $P(\Gamma)$ consist of *all* f in $C(\Gamma)$ which are boundary function of functions analytic in W? Walsh [2] in 1926 showed

An address delivered before the New York meeting of the Society on February 25, 1961 by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors March 30, 1962.

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