FIBERINGS OF SPHERES AND H-SPACES WHICH ARE RATIONAL HOMOLOGY SPHERES

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There are various fiber bundles known whose total spaces are spheres. Examples are the Hopf maps $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$ and and fiberings over projective spaces. It is an open question whether these are all the fiberings of spheres with connected fibers.

Spanier and Whitehead [6] showed that in such a fibering the fiber must be an H-space. The results of Borel [2; 3] showed that the fiber is a rational homology sphere.

THEOREM 1. *Let* $p: S^n \to B$ *be a fiber bundle map onto a polyhedron* B, *with fiber* F *a connected polyhedron. Then* F *is the homotopy type of* S^1, S^3 *or* S^7.

The proof depends on studying the torsion of $H^*(F)$, in particular the 2-torsion.

THEOREM 2. *Let* X *be an* H-*space, connected, with* $H_i(X)$ *finitely generated for all* i, *zero for large* i. *Suppose in addition that* X *is a rational homology sphere, i.e.,* $H_*(X; Q) \cong H_*(S^n; Q)$ *for some* n. *If* $H^1(X; Z_2) = 0$, *then*

$$H^*(X; Z_2) = \Lambda(x) \otimes Z_2[Sq^1 x]/(Sq^1 x)^{2^q}, \quad 0 \le q < \infty,$$

dim x *is odd*, > 1.

If $H^1(X; Z_2) \ne 0$, *then* $H^*(X; Z_2) = Z_2[x]/x^{2^q}$, $1 \le q < \infty$, dim $x = 1$.

Since F satisfies the hypothesis of Theorem 2, we may apply it to this situation. A spectral sequence argument using Theorem 2, similar to Borel's argument [3, p. 165], yields the result that $H^*(F; Z_2) = \Lambda(x)$, i.e., F is a mod 2 homology sphere. Namely, Theorem 2 shows that $H^*(F; Z_2)$ has a simple system of transgressive generators, and employing a theorem of Borel [3, Proposition 16.1], we get that $H^*(B; Z_2)$ is a polynomial ring on the "transgressions" of the generators, in dimensions $< n$. But now, analyzing the structure of the spectral sequence, we cannot arrive at $E_\infty \cong H^*(S^n; Z_2)$, unless $H^*(F; Z_2)$ has only one generator; i.e., $H^*(F; Z_2) = \Lambda(x)$.

Adams [1] has shown that a mod 2 homology sphere which is an H-space is a 1, 3, or 7 dimensional mod 2 homology sphere, hence a rational homology 1, 3, or 7 sphere. It is then easy to show that an H-space which is a rational homology 1, 3, or 7 sphere has no odd tor-