SOME TWO-GENERATOR ONE-RELATOR NON-HOPFIAN GROUPS

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In 1951 Graham Higman claimed (in [1]) that every finitely generated group with a single defining relation is Hopfian,[2] attributing this fact to B. H. Neumann and Hanna Neumann. However we shall show that this is not, in any way, the case. For example the group

$$G = gp(a, b;\ a^{-1}b^2a = b^3) \tag{1}$$

is non-Hopfian. Hence the following question of B. H. Neumann [2, p. 545] has a *negative* answer: *Is every two-generator non-Hopfian group infinitely related?*

This group G turns out to be useful for deciding a somewhat different kind of question. For Graham Higman[3] has pointed out that G can, of course, be generated by a and b^4. However it transpires that in terms of these generators G requires *more than one relation* to define it. Thus Higman has produced a counter-example to the following well-known conjecture: *Let G be generated by n elements $a_1, a_2, \cdots, a_n$ and let r be the least number in any set of defining relations between $a_1, a_2, \cdots, a_n$. Then $n-r$ is an invariant of G (i.e. does not depend on the particular basis $a_1, a_2, \cdots, a_n$).* This conjecture has received some attention in the past; indeed there is a "proof" of it by Petresco [3].

The group defined by (1) is clearly only one of a larger family of groups of the kind

$$G = gp(a, b;\ a^{-1}b^l a = b^m). \tag{2}$$

It is convenient at this point to introduce a definition. Thus we say two nonzero integers l and m are meshed if either

(i) l or m divides the other,

or,

(ii) l and m have precisely the same prime divisors. This definition enables us to distinguish easily between the Hopfian and the non-Hopfian groups in the family of groups (2). For the following theorem holds.

THEOREM 1. *Let l and m be nonzero integers. Then*

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[2] A group G is Hopfian if $G/N \cong G$ implies $N=1$; otherwise G is non-Hopfian.

[3] In a letter.