RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

SUPPORTS OF A CONVEX FUNCTION[1]

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Let C be a real, symmetric, $m\times m$, positive-semi-definite matrix. Let $R^m=\{(x_1, \cdots, x_m)\mid x_i$ is a real number, $i=1, \cdots, m\}$, and let $K\subset R^m$ be a polyhedral convex cone, i.e., there exists a real $m\times n$ matrix A such that $K=\{x\mid x\in R^m$ and $xA\leqq 0\}$. Consider the function $\psi: K\to R$ defined by $\psi(x)=(xCx^T)^{1/2}$ for all $x\in K$. We wish to characterize the set, U, of all supports of ψ, where

$$(1)\qquad U = R^m \cap \{u \mid x\in K \Rightarrow ux^T \leqq (xCx^T)^{1/2}\}.$$

Let $R^n_+=R^n\cap\{\pi\mid \pi\geqq 0\}$ and consider the set

$$(2)\ V = \{v\mid \exists x\in R^m, \pi\in R^n_+ \text{ and } v=\pi A^T+xC, xCx^T\leqq 1, \quad xA\leqq 0\}.$$

We shall demonstrate:

THEOREM. $U=V$.

We first show:

LEMMA 1. $x, y\in R^m \Rightarrow (xCy^T)^2\leqq(xCx^T)(yCy^T)$.

PROOF. If $x, y\in R^m$ consider the polynomial $p(\lambda)=\lambda^2xCx^T+2\lambda xCy^T+yCy^T=(x+\lambda y)C(x+\lambda y)^T$. Since C is positive-semi-definite, $p(\lambda)\geqq 0$ for all real numbers λ, and thus the discriminant of p is nonpositive, i.e.,

$$4(xCy^T)^2 - 4(xCx^T)(yCy^T) \leqq 0. \qquad \text{q.e.d.}$$

As an immediate application of Lemma 1 we show:

LEMMA 2. $V\subset U$.

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