HOLOMORPHIC FIBER BUNDLES OVER RIEMANN SURFACES

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For the purpose of this paper a fiber bundle $F \to X$ over a Riemann surface X is meant to be a fiber bundle in the sense of N. Steenrod [62] where the base space is X, the fiber a complex space, the structure group G a complex Lie group that acts as a complex transformation group on the fiber, and the transition functions $g_{ij}(x)$ are holomorphic mappings into G. Correspondingly, cross-sections are assumed to be holomorphic cross-sections. We shall use freely the notations of [62]. Whenever we report about families of fiber bundles we mean holomorphic families of fiber bundles; the basic notations concerning families of fiber bundles can be found in [30] and shall also be used freely. Triviality of bundles resp. families of bundles is always supposed to be holomorphic triviality.

1. Classification of fiber bundles and reduction of the structure group. The classification of fiber bundles over noncompact Riemann surfaces offers no problem on account of

THEOREM 1.0 [15; 54]. *Every fiber bundle over a noncompact Riemann surface is trivial, provided the structure group G is connected.*[1]

For compact Riemann surfaces, however, the situation is entirely different. In the general case it seems to be quite difficult to give a classification. Yet one has some results if either the fiber and the structure group or else the base space is sufficiently special.

For the rest of this section X shall always denote a compact Riemann surface unless stated otherwise.

There is a preliminary result concerning line bundles $L \to X$, i.e. fiber bundles with fiber the complex line C^1 and structure group the multiplicative group $GL(1, C)$ of complex numbers acting upon C^1 in the usual way, a result that first has been proved in a much more

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[1] If G is not connected, the theorem fails to be true as can be seen from the following example: Let X be the complex plane without the origin, G the multiplicative group of nth roots of unity acting on the unit circle by (left) multiplication, and the fiber bundle be defined by a covering of X, consisting of two domains U_1, U_2 bounded by straight lines through the origin, and the transition function that equals 1 in one component of $U_1 \cap U_2$ and $e^{2\pi i/n}$ in the other component of $U_1 \cap U_2$.