ON THE EXISTENCE OF A SMALL CONNECTED OPEN SET WITH A CONNECTED BOUNDARY

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Communicated by Edwin Moise, December 23, 1961

In a connected, locally connected, locally compact metric space with no local separating point it is rather easy to construct an arbitrarily small connected open set whose boundary is a subset of an arbitrary small continuum lying in its complement. In fact such sets form a topological basis for the space. However, it seems to be much more difficult to construct small connected open sets whose boundaries are connected. The author constructed such open sets (substituting something weaker for local compactness) [1, Theorem 33] in certain special plane-like spaces but efforts at that time to generalize the theorem failed. Now with the help of the partitioning technique (brick partitioning, in particular) the construction may be carried out successfully.[2]

LEMMA. *Suppose that* (1) *U is a connected open proper subset of the connected, locally connected, compact metric space M such that $\overline{U}=M$,* (2) *p is a point of U such that $M-p$ is connected and* (3) *no point of $M-p$ is a local separating point of M. Then if ϵ is a positive number, there exists a connected open point set V such that* (1) *$p\in V\subset\overline{V}\subset U$,* (2) *$M-\overline{V}$ is connected and* (3) *if $x\in M-V$, $d(x, V)<\epsilon$.*

INDICATION OF PROOF. Let F denote the boundary of U. Without loss of generality we shall assume that 3ϵ is less than $d(p, F)$. Being compact and locally connected, M has property S. By Theorem 8 of [2] there exists a sequence $G_1, G_2, \cdots$ such that G_i is a brick $(1/i)$-partitioning of M and G_{i+1} is a refinement of G_i. Let B_i denote the subcollection of elements g of G_i such that $\bar{g}\cdot F\neq 0$ and let H_i denote the subcollection of G_i consisting of the elements of B_i together with all other elements of G_i which are separated from p by $\overline{B}_i^*$.

There exists a value of i such that each point of $\overline{H}_i^*$ is within $\epsilon/4$ of F. For suppose on the contrary that for each i, $\overline{H}_i^*$ contains a point q_i such that $d(q_i, F)\geqq\epsilon/4$. Let us suppose that $\{q_i\}$ converges to q (for certainly some subsequence converges). Since $d(q, F)\geqq\epsilon/4$, q belongs to U and there is an arc pq from p to q lying in U. Now let $i(q)$ be a value of i large enough so that if $g_1, g_2\in G_i$, $\bar{g}_1\cdot pq\neq 0$ and $\bar{g}_2\cdot F\neq 0$,

[1] Work done under NSF G-9418.

[2] I wish to thank Professor A. H. Stone for calling my attention to the fact that the problem was still unsolved.