THE WAVE EQUATION IN EXTERIOR DOMAINS

BY PETER D. LAX[1] AND RALPH S. PHILLIPS[2]

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This note deals with solutions of the wave equation in three dimensions in the exterior of a finite number of smooth obstacles, on whose boundaries the solution is subject to boundary conditions of the form $u=0$ or $u_n=\sigma u$, σ a non-negative function. We shall show that every such solution of finite energy propagates eventually out to infinity and behaves asymptotically like a free space solution.

THEOREM I. *For every nonzero solution there exists a positive constant d less than the total energy of the solution such that given any bounded domain, there is a time at which the energy contained in the exterior of this domain exceeds d.*

SKETCH OF PROOF. Suppose the theorem is false for some u; then given any positive ϵ there exists a bounded domain such that the energy contained in its exterior is less than ϵ for all time. Assume that u is a smooth function;[3] applying the law of conservation of energy to u_t and using standard estimates for u_{xx} in terms of Δu, we see that the square integral of the second partial derivatives of u over the exterior of the obstacles is uniformly bounded for all time. Let $\{t_n\}$ be an arbitrary sequence of numbers; using the Rellich selection theorem[4] and a diagonal process we can choose a subsequence such that the first partial derivatives of u form a Cauchy sequence in the square integral sense over any bounded domain in x-space. Since we have supposed that the energy contained outside bounded domains is uniformly small for all t, it follows that $\{u(t_n)\}$ is a Cauchy sequence in the energy norm over the whole exterior. This shows that $u(t)$ is a vector-valued almost periodic function of t. By the main theorem of a.p. functions $u(t)$ is a superposition of exponentials:

$$u \approx \sum a_j(x)e^{i\omega_j t}.$$

From the mean value expression for the coefficient a_j it follows that it is a solution of the reduced wave equation $\Delta a+\omega_j^2 a=0$. But according to a theorem of Rellich,[5] the reduced wave equation has for $\omega_j\neq 0$

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[3] u can be made smooth by mollifying it with respect to t.

[4] F. Rellich, Göttinger Nachrichten, 1930.

[5] F. Rellich, Jber. Deutsch. Math. Verein. **53** (1943), 57–65.