AN INEQUALITY FOR LINEAR TRANSFORMATIONS WITH EIGENVALUES

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The purpose of this announcement is to state theorems concerning bounded linear transformations on Hilbert space which are far more general than the recent theorems of H. D. Block and W. H. J. Fuchs [2]. Our theorems are more general even in the case that the transformation is a matrix, as in [2]. The basic idea involved in these theorems was first communicated to the author by Professor H. F. Bohnenblust in 1957, and has been applied meanwhile to various perturbation problems for ordinary and partial differential equations, [1; 4]. The theorems in essentially their present form were enunciated by the author in an unpublished manuscript sent to Professor Bohnenblust in July, 1960.

THEOREM 1. *Let T be a bounded linear transformation on Hilbert space $\mathfrak{H}$ having a complete orthonormal set of eigenelements y_i. Let α be an arbitrary complex number and let ϵ be a nonnegative real number. Let $P(\epsilon)$ be the projection operator from $\mathfrak{H}$ onto the subspace $\mathfrak{F}(\epsilon)$ spanned by all the y_i whose corresponding eigenvalues λ_i lie in the open disk $|\lambda-\alpha|<\epsilon$. Then for any $x\in\mathfrak{H}$,*

$$\|Tx - \alpha x\| \geqq \epsilon\|x - P(\epsilon)x\|. \tag{1}$$

PROOF. By hypothesis, $Ty_i=\lambda_i y_i$ and $(y_i, y_j)=\delta_{ij}$. It is easily verified that $(Tx-\alpha x, y_i)=(\lambda_i-\alpha)(x, y_i)$, and hence the Parseval completeness formula gives

$$\|Tx - \alpha x\|^2 = \sum_i |\lambda_i - \alpha|^2 |(x, y_i)|^2 \geqq \epsilon^2 \sum_i^* |(x, y_i)|^2,$$

where the * denotes summation over only those indices i for which $|\lambda_i-\alpha|\geqq\epsilon$. The result (1) is then immediate.

The following theorem is a corollary of Theorem 1.

THEOREM 2. *Let T be a symmetric, completely continuous linear transformation on Hilbert space $\mathfrak{H}$. Let α and ϵ be real numbers with ϵ nonnegative. Let $P(\epsilon)$ be the projection operator onto the subspace $\mathfrak{F}(\epsilon)$ spanned by all the eigenelements of T whose corresponding eigenvalues lie in the interval $(\alpha-\epsilon, \alpha+\epsilon)$. Then for any $x\in\mathfrak{H}$, the inequality (1) is valid.*