ORIENTABLE SURFACES IN FOUR-SPACE

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1. Introduction. Fränkl and Pontrjagin [1] and Seifert [2] have shown that for any given family of disjoint polyhedral simple closed curves in three-space, there can always be found a polyhedral orientable surface in three-space whose boundary consists precisely of the given curves. The following theorem extends this result to surfaces in four-space.

THEOREM 1. *Let M^2 be a locally flat, polyhedral, closed orientable surface (not necessarily connected) in Euclidean four-space, R^4. Then there is an orientable polyhedral three-manifold, M^3, in R^4, whose boundary is M^2.*

Local flatness means that for each vertex v of M^2, the link of v on M^2 (a simple closed curve) is unknotted in the link of v in R^4 (a three-sphere). This condition is purely local and absolutely necessary. On the other hand, the restriction to orientable surfaces is required by the nature of the proof, and I do not know whether nonorientable surfaces of even characteristic in four-space bound nonorientable three-manifolds in four-space.[2]

2. Outline of the proof. M^2 is first deformed so that its intersections with the horizontal hyperplanes $R_t^3 = \{(x_1, x_2, x_3, x_4) : x_4 = t\}$ are as simple as possible. What we have in mind is to find orientable surfaces in the R_t^3 whose boundaries are precisely $M^2 \cap R_t^3$, in such a continuous way that when considered together they form an orientable three-manifold M^3 whose boundary is M^2. The process is carried out with decreasing t, and the local flatness of M^2 assures us that the construction can be begun. As t decreases, $M^2 \cap R_t^3$ changes isotopically, except at a finite number of singular values of t. A slight deformation of M^2 insures that we need only consider hyperbolic transformations, in which two arcs come together at a midpoint and then separate like the cross-sections of a saddle surface, and elliptic transformations, in which a simple closed curve shrinks to a point and then disappears (or vice versa). In the hyperbolic case, these arcs already form part

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[2] *Added in proof.* This case is considered in a forthcoming paper.