THE EMBEDDING OF TWO-SPHERES IN THE FOUR-SPHERE

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1. **Introduction.** An embedding h, of S^2 in S^4 will be called *smooth* if h can be extended to an embedding of $S^2 \times E^2$ in S^4, where E^2 is the open unit disc in the plane and S^2 is identified with $S^2 \times$origin. The embedding h will be called *semi-linear* if h is a simplicial homeomorphism of some rectilinear subdivision of the boundary of a 3-simplex with a subcomplex of some rectilinear subdivision of the boundary of a 5-simplex. If S^2 is semi-linearly embedded in S^4 and v is a vertex of S^2, then the link of v on S^2 is a simple closed curve, while the link of v in S^4 is a three-sphere. If this simple closed curve is unknotted in this three-sphere, then S^2 is said to be *locally flat at v.* If $S^2 \subset S^4$ is locally flat at all of its vertices, then it is said to be *locally flat.* Noguchi has shown in [1, Theorem 3] that every locally flat semi-linear S^2 in S^4 is smoothly embedded, and that its regular neighborhoods are homeomorphic to $S^2 \times E^2$.

Let S^2 have a neighborhood $S^2 \times E^2$ in S^4. Let $E^{2\prime}$ be the open disc of radius 1/2. Then $A^4 = S^4 - (S^2 \times E^{2\prime})$ is an orientable four-manifold whose single boundary component is homeomorphic to $S^2 \times S^1$. A^4 is called an *exterior* of S^2 in S^4. If the boundary component of A^4 is removed, the remaining open manifold is clearly homeomorphic to $S^4 - S^2$. I do not know whether two different exteriors of a smooth S^2 in S^4 are necessarily homeomorphic. In the locally flat semi-linear case, however, we can define the exterior to be the complement in S^4 of any open regular neighborhood of S^2. Then *the* exterior is well-defined, for one can always find a semi-linear homeomorphism of S^4 onto itself taking one regular neighborhood of S^2 onto any other (see [2]).

Two pairs (S^4, S^2) and $(S^{4\prime}, S^{2\prime})$ are said to be *equivalent* (or homeomorphic) if there is a homeomorphism h of S^4 onto $S^{4\prime}$ which carries S^2 onto $S^{2\prime}$. We can then in any case ask how strong an invariant of the pair (S^4, S^2) we get by considering the topological type of an exterior A^4 of S^2 in S^4. In other words, how many nonequivalent smooth embeddings of S^2 in S^4 can have an exterior homeomorphic to A^4?

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