ARITHMETIC SUBGROUPS OF ALGEBRAIC GROUPS

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A complex algebraic group G is in this note a subgroup of $\boldsymbol{GL}(n, \boldsymbol{C})$, the elements of which are all invertible matrices whose coefficients annihilate some set of polynomials $\{P_\mu[X_{11}, \cdots, X_{nn}]\}$ in n^2 indeterminates. It is said to be defined over a field $K \subset \boldsymbol{C}$ if the polynomials can be chosen so as to have coefficients in K. Given a subring B of $\boldsymbol{C}$, we denote by G_B the subgroup of elements of G which have coefficients in B, and whose determinant is a unit of B. Assume in particular G to be defined over $\boldsymbol{Q}$. Then $G_{\boldsymbol{Z}}$ is an "arithmetically defined discrete subgroup" of $G_{\boldsymbol{R}}$, or, more briefly, an *arithmetic subgroup* of $G_{\boldsymbol{R}}$. A typical example is the group of units of a nondegenerate integral quadratic form, and as a matter of fact, the main results stated below generalize facts known in this case from reduction theory. The proofs will be published elsewhere.

1. **Reductive groups.** A complex algebraic group G is an *algebraic torus* (a torus in the terminology of [1]) if it is connected and can be diagonalized or, equivalently, if it is birationally isomorphic to a product of groups $\boldsymbol{C}^*$ [1, Chapter II]. The group G is *reductive* if its identity component G^0 may be written as $G^0 = T\cdot G'$, where T is a central algebraic torus, and G' is an invariant connected semi-simple group, or, equivalently, if all rational representations of G are fully reducible.

LEMMA 1. *Let* $G_1 \supset \cdots \supset G_m$ *be reductive algebraic subgroups of* $\boldsymbol{GL}(n, \boldsymbol{C})$, *defined over* $\boldsymbol{R}$. *Then there exists* $a \in \boldsymbol{SL}(n, \boldsymbol{R})$ *such that the groups* $a \cdot G_{i\boldsymbol{R}} \cdot a^{-1}$ *are stable under* $x \rightarrow {}^t x$ $(i = 1, \cdots, m)$.

This lemma, formulated in a somewhat different terminology, is due to G. D. Mostow [4]. Lemma 1, for $m = 1$, implies easily that the usual properties of maximal compact subgroups and of the Iwasawa decompositions (see [7] for instance) are valid for real algebraic reductive groups.

LEMMA 2. *Let* G *be a connected reductive complex algebraic group,* H *an algebraic subgroup. Then* G/H *is an affine variety if and only if* H *is reductive. If* G *and* H *are defined over* $\boldsymbol{Q}$, *and* H *is reductive, there exists a rational representation* $\pi: G \rightarrow \boldsymbol{GL}(m, \boldsymbol{C})$, *defined over* $\boldsymbol{Q}$, *such*