THE THEORY OF TRANSFINITE RECURSION[1]

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During recent years there has been growing interest in logical calculi with infinitely long expressions (see e.g. [2; 4; 5]). However, all results obtained so far, many of them very remarkable, are semantic in character. The reason is that, while the syntax of the "classical" calculi could be "Gödelized" and then studied with the aid of the theory of recursive functions, no such procedure has so far been devised for the infinitistic calculi. To do this, one should possess transfinite analogues of the theory of recursive functions and of arithmetization.

In the investigations reported here, such analogues are constructed and applied to infinitistic calculi.

Let ω_α be an arbitrary, but fixed, regular initial ordinal. By *ordinals* we shall mean ordinals $<\omega_\alpha$, by a *sequence*—a well ordered sequence similar to an ordinal, by the *length* of a sequence—its order type, and by a *function*—a function whose (single) argument ranges over the set of all sequences (of a fixed length) of ordinals and whose values are ordinals.

ω_α-recursiveness. Our first aim is to explicate the notion *a function whose value, for each given value of its argument, is "calculable" in a sequence of steps.* This is done with the aid of a formalism (with *infinitistic* rules of formation and transformation) which is analogous to Kleene's formalism for recursive functions. (See [3, Chapter XI].) We use much of the metamathematical terminology of [3], in a sense analogous to that in which it is used there.)

PRIMITIVE SYMBOLS. "$=$", "$'$" (stroke), "sup" (the supremum operator), "0", a variable "x_ξ" for each $\xi<\omega_\alpha$, a function letter (f.l.) "f_ξ" for each $\xi<\omega_\alpha$.

NUMERALS. 0 followed by a sequence of length β of strokes is the numeral for β. We denote it by $\boldsymbol{\beta}$.

TERMS. (a) Each numeral is an (atomic) term.

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