STRONG RATIO LIMIT PROPERTY

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1. **Introduction.** For every nonnegative integer n let $p_{ij}^{(n)}$ be the n-step transition probabilities of a recurrent, irreducible, aperiodic Markov chain, $i, j=0, 1, \cdots$. We say the chain has the *strong ratio limit property* (*SRLP*) if there exist positive constants $\pi_j, j=0, 1, \cdots$, such that

$$(1) \qquad \lim_{n\to\infty} \frac{p_{ij}^{(n+m)}}{p_{kh}^{(n)}} = \frac{\pi_j}{\pi_h}, \qquad m = 0, \pm 1, \pm 2, \cdots.$$

It is well known that SRLP does not hold for all chains of the type considered here.[2] We here present conditions for SRLP; the continuous parameter case is also considered.

2. **Discrete parameter.** Let ${}_kp_{ij}^{(n)}=$ Prob [going from i to j in n steps without visiting k at step number $1, 2, \cdots, n-1$]. Note

$$(2.1) \qquad \sum_{n=1}^{\infty} {}_jp_{ij}^{(n)} = 1 \quad \text{and} \quad \text{g.c.d.}\{n: p_{ii}^{(n)} > 0\} = 1 \quad \text{for every } i, j.$$

LEMMA 1. SRLP *holds if and only if* $p_{00}^{(n+1)}/p_{00}^{(n)} \to 1$ *as* $n\to\infty$.

SKETCH OF PROOF. Assume $p_{00}^{(n+1)}/p_{00}^{(n)} \to 1$ as $n\to\infty$. For $n>N\geq 1$ we have

$$(2.2) \qquad \begin{aligned} A_n(\alpha) &= p_{0\alpha}^{(n)}/p_{00}^{(n)} = \sum_{v=1}^{n} {}_0p_{0\alpha}^{(v)} p_{00}^{(n-v)}/p_{00}^{(n)} = \sum_{v=1}^{N} + \sum_{v=N+1}^{n} \\ &= B_{N,n}(\alpha) + C_{N,n}(\alpha). \end{aligned}$$

Observe $A_n(\alpha)$ converges as $n\to\infty$ if and only if

$$(2.3) \qquad \lim_{N\to\infty} \lim_{n\to\infty} C_{N,n}(\alpha)$$

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[2] For counterexample, general discussion and references to the literature see [1] under "ratio limit theorem, individual."

Frequently authors consider only the case $m=0$ in relation to (1). We do not know whether this is really more restrictive or not.