OBSTRUCTIONS TO THE EXISTENCE OF ALMOST COMPLEX STRUCTURES

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1. **Definitions and notation.** Let M be an orientable, differentiable manifold of dimension $2n$ and let $\xi=(E_\xi, M, R^{2n}, \pi)$ denote the tangent bundle of M; we assume the structural group of ξ has been reduced from the full linear group to the special orthogonal group $SO(2n)$. By definition, M admits an almost complex structure if and only if the associated fibre bundle $\eta=(E, M, \Gamma_n, p)$ admits a cross section;[1] here Γ_n denotes the homogeneous space $SO(2n)/U(n)$. In this paper, we will study the obstructions to a cross section for any fibre bundle $\theta=(E, B, \Gamma_n, p)$ with structural group $SO(2n)$ and base space B a CW-complex. If $s: B^q \to E$ is a cross section of θ over the q-skeleton of the base space B, then the obstruction to extending s over the $(q+1)$-skeleton is denoted by

$$c^{q+1}(s) \in H^{q+1}(B, \pi_q(\Gamma_n)).$$

Since θ is a bundle with structural group $SO(2n)$, the following characteristic classes are defined:

(a) Integral Stiefel-Whitney classes,

$$W_i(\theta) \in H^i(B, Z), \qquad 3 \leqq i \leqq 2n-1, \qquad i \text{ odd}.$$

(Recall that $2 \cdot W_i(\theta)=0$.)

(b) Euler-Poincaré class, $W_{2n}(\theta) \in H^{2n}(B, Z)$.

(c) Pontrjagin classes $p_i(\theta) \in H^{4i}(B, Z)$, $0 \leqq i \leqq n$.

In an analogous manner, if ξ is a fibre bundle with base space B and structural group $U(n)$, the Chern classes of ξ will be denoted by $c_i(\xi) \in H^{2i}(B, Z)$, $0 \leqq i \leqq n$.

2. **Statement of results.** The homotopy group $\pi_q(\Gamma_n)$ is called *stable* if $q<2n-1$; it is well known that the stable homotopy groups $\pi_q(\Gamma_n)$ for fixed q and variable n are all isomorphic; see Gray [4, p. 432]. The stable homotopy groups of Γ_n have been determined by Bott [2]; he showed that in the stable range,

[1] Standard references on the subject of almost complex structures are Ehresmann's lecture at the 1950 International Congress of Mathematicians [3] and the last section of Steenrod's book [10].

The author would like to take this opportunity to acknowledge that his proof of the two theorems announced in Abstract 60T-24, Notices Amer. Math. Soc. vol. 7 (1960) p. 1001, contains an apparently irreparable gap. Whether or not these two theorems are correct is not known.