ON THE EIGENVALUES OF POSITIVE OPERATORS[1,2]

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The classical theory of Frobenius-Perron concerning the distribution of eigenvalues of a matrix with non-negative elements has been variously extended to positive operators, that is, linear operators on function spaces transforming non-negative functions into non-negative functions. Since the classical work of Jentzsch (see bibliography), there have been two kinds of extensions: (a) it has been established under various conditions that a positive operator shall have a positive eigenvector with positive eigenfunction (see e.g. Birkhoff [1], Karlin [4], Samelson [7], Schaefer [8]), and (b) attempts have been made to extend the Frobenius theorem stating that, for a non-negative matrix with spectral radius one, the eigenvalues on the unit circle are roots of unity. This result has been extended to positive operators under strong additional assumptions, all of them guaranteeing that the intersection of the spectrum with the unit circle shall consist of isolated points only. Results of this kind are summarized in Karlin. It is our present purpose to establish the second result under fairly general conditions. Our result is the following:

THEOREM. *Let P be a linear operator of norm at most one (a "contraction") in $L_1(S, \Sigma, \mu)$, where (S, Σ, μ) is a measure space of finite measure. Suppose that P is bounded, with norm at most one, in $L_\infty(S, \Sigma, \mu)$. Then:*

If α is an eigenvalue of P, and $Pf = \alpha f$ for some nonzero integrable f, then α^2 is also an eigenvalue of P.

If f is written in the form $f = |f| g$, where the function g is of absolute value one at every point, then the eigenfunction belonging to α^2 is $|f| g^2$.

Operators of the type described in this theorem, the so-called dissipative operators, are commonly encountered in various circumstances, for instance in the study of semigroups generated by second-order differential operators. The property of contractivity in L_1 is also crucially assumed in the pointwise ergodic theorems.

The proof of this theorem can be based upon elementary functional-analytic techniques, and will be given in full here. We begin by making the following two simplifications: (a) It can be assumed

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