DERIVATIONS AND GENERATIONS OF FINITE EXTENSIONS

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Let k be a given ground field, let $\mathfrak{F}_r$ denote the class of finite (=finitely generated) field extensions of k of tr.d. (=transcendence degree) $\leqq r$, and let n be the function defined on $\mathfrak{F}=\bigcup_0^\infty \mathfrak{F}_r$ by: for any $L\in\mathfrak{F}$, $n(L)$ = the minimal number of generators of L/k. Classically it is known for suitable k that there exist purely transcendental extensions L/k having tr.d. 2, and containing impure subextensions of tr.d. 2, a fact which shows that in general n is not monotone in $\mathfrak{F}$ for all k. The main result of this note establishes that these "counterexamples to Lüroth's theorem" constitute the only barriers to the monotonicity of n (see Theorem 2 for a precise statement). In particular it is demonstrated that n is montone on $\mathfrak{F}_1$ for arbitrary k, a result which appears new even when restricted to the subclass $\mathfrak{F}_0$ of finite algebraic extensions of k.

A result of independent (and possibly more general) interest, which is proved below, and which is essential to our proof of the statements above, is that dim $\mathfrak{D}$ is montone on $\mathfrak{F}$, where for any $L\in\mathfrak{F}$, $\mathfrak{D}(L)$ is the vector space over L of k-derivations of L. The connection between n and dim $\mathfrak{D}$ is given in the lemma.

LEMMA. *Let L/k be a finite extension of tr.d. r, let $s=\dim \mathfrak{D}(L)$, and let $n=n(L)$. Then $s\leqq n\leqq s+1$; if $s>r$, then $n=s$.*[2]

PROOF. It is known (e.g. [3, Theorem 41, p. 127]) that s is the smallest natural number[3] such that there exist elements $u_1, \cdots, u_s\in L$ such that L is separably algebraic over the field $U=k(u_1, \cdots, u_s)$. Then $L=U(a)$ for some $a\in L$, so that $s\leqq n\leqq s+1$.

If $s>r$, there exists u_q in the set $S=\{u_1, \cdots, u_s\}$ such that u_q is algebraically dependent over k on the complement of u_q in S. For convenience renumber so that u_s is algebraic[4] over the field $T=k(u_1, \cdots, u_{s-1})$. A short argument shows that $L=U(a)$ for some

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[2] Expressed in the other words: If L/k is not separably generated, then $n(L)$ $=\dim \mathfrak{D}(L)$.

[3] Strictly speaking the notation should allow for the case $s=0$. By agreement then $U=k$.

[4] In case $s=1$ set $T=k$.