A NEW PROOF AND AN EXTENSION OF HARTOG'S THEOREM[1]

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Let R denote n dimensional real euclidean space and let Ω_0 be a shell in R, by this we mean that there exist open sets Ω_1, Ω_2 where Ω_1 is relatively compact and has its closure contained in Ω_2, and $\Omega_0 = \Omega_2 -$ closure Ω_1. Call Γ_j the boundary of Ω_j. Let $D=(D_1, \cdots, D_r)$ be a sequence of linear partial differential operators with constant coefficients on R with $r>1$. For a function f on R we write $Df=0$ if $D_jf=0$ for $j=1, 2, \cdots, r$. We want to determine the conditions on D in order that the following property should hold: If f is an indefinitely differentiable function on Ω_0 with $Df=0$ then there exists a unique indefinitely differentiable function h on Ω_2 with $Dh=0$ and $h=f$ on Ω_0. Hartog's theorem asserts that such an extension of f is possible if R is complex euclidean space of complex dimension $n/2=m>1$ and Ω_1 and Ω_2 are topological balls, and $D_j=\partial/\partial x_{2j-1}+i\,\partial/\partial x_j$ for $j=1, 2, \cdots, m$ where $x=(x_1, \cdots, x_n)$ are the coordinates on R. An extension of Hartog's theorem has been found by S. Bochner in [1] by a different method.

We can find a function g defined and C^∞ on Ω_2 such that $g=f$ on Ω_0 except on an arbitrarily small neighborhood $N(\Gamma_1)$ in Ω_0. (We choose $N(\Gamma_1)$ so small that its closure does not meet Γ_2.) Call $\Omega_3=\Omega_1\cup N(\Gamma_1)$. We have $Dg=0$ on $\Omega-\Omega_3$. We set $g_j=D_jg$, so g_j are C^∞ and have their supports in the closure of Ω_3; in particular the g_j are of compact support. For any j, k,

$$D_kg_j = D_jg_k \tag{1}$$

since both sides are equal to D_kD_jg in Ω_3 and zero outside.

Next we take the Fourier transforms: Call P_k the Fourier transform of D_k and G_k that of g_k; P_k is a polynomial and G_k an entire function of exponential type on C (complex n-space); the exponential type of G_k is determined by the convex hull K of Ω_3. Moreover, G_k decreases on the real part of C faster than the reciprocal of any polynomial (see [5]). Relation (1) becomes

$$P_k(z)G_j(z) = P_j(z)G_k(z). \tag{2}$$

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