A CLASS OF CONFORMAL METRICS

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1. Conformal metrics of curvature no greater than -4 are important in certain function-theoretic questions. Such metrics were introduced by Ahlfors [1] in his paper in which it was shown that the Bloch constant was at least as large as $3^{1/2}/4$. Ideas of this paper were applied to the Picard theorem and allied questions by R. M. Robinson [3]. The present communication summarizes the investigations of the author in which it is shown that the differential conditions appearing in Ahlfors's work may be replaced by conditions involving upper semi-continuity and mean-value properties. In this recasting the theory parallels that of subharmonic functions. The methods of Perron are available and serve as a basis for existence theorems for conformal metrics of constant curvature -4. The results developed in the present investigations have a number of function-theoretic consequences. We cite two. The first is that the Bloch constant exceeds $3^{1/2}/4$. The second is the following theorem:

Suppose that B is the class of nonconstant bounded analytic functions f on $\{|z|<1\}$ and that $n_1(r;f)$ denotes the number of zeros of f', counted according to multiplicity, in $\{|z|<r\}$, $0<r<1$. Then we have

$$\sup_{f\in B}\left\{\limsup_{r\to 1}\frac{\int_{r_0}^{r} t^{-1}n_1(t;f)\,dt}{-\log(1-r)}\right\}=1,\qquad 0<r_0<1. \tag{1.1}$$

2. **S-K metrics.** Let F be a Riemann surface and let S denote the family of uniformizers defining the conformal structure on F. By a *conformal metric* on F we mean a function λ with domain S assigning to $\sigma\in S$ a non-negative finite-valued function λ_σ whose domain is that of σ such that, if σ, $\tau\in S$ have images with nonempty intersection O and $\theta=\{(\sigma^{-1}(p),\tau^{-1}(p))\mid p\in O\}$, then $\lambda_\sigma(z)=\lambda_\tau[\theta(z)]\,|\theta'(z)|$, $z\in$domain of θ. If ϕ is a uniformizer, there exists a unique extension of λ to $S\cup\{\phi\}$ which is a conformal metric in the sense of the conformal structure $S\cup\{\phi\}$. We let λ_ϕ denote the image of ϕ with respect to λ so extended. The notion of upper semi-continuity of λ at a point $p\in F$ is referred to that of λ_σ ($p\in$image of σ) at the point $\sigma^{-1}(p)$. Similar remarks hold for differentiability properties, inequality, and vanishing. By an *S-K metric* is meant a conformal metric on F which is upper semi-continuous at each point of F and satisfies the condition